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Positive speed of tagged particle with $\pm 1, 2$ jumps in symmetric exclusion process on $\mathbb{Z}$

מהירות חיובית של חלקיק מסומן עם קפיצות של $\pm 1, 2$ צעדים בתהליך הפרדה סימטרי על שריג השלמים

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POSITIVE SPEED OF TAGGED PARTICLE WITH $\pm 1, 2$ JUMPS IN SYMMETRIC EXCLUSION PROCESS ON $\mathbb{Z}$
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Abstract. We prove that the position of the tagged particle X_t for the modified exclusion process on $\mathbb{Z}$, in which the tagged particle jumps $\pm 1, 2$ steps with rate $\frac{1}{4}$ while the other particles jump ± 1 steps with rate $\frac{1}{2}$, satisfies $\frac{X_t}{\sqrt(t)}$ converges in distribution to a non-degenerate Gaussian random variable with zero mean.

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1. Introduction

We observe the trajectory of the particle which at time 0 is located at the origin in the exclusion process . We call this particle the tagged particle. Let X_t denote the position of the tagged particle in the exclusion process on the lattice $\mathbb{Z}^d$ with initial distribution given by $\overline{\nu_\alpha} = \nu_\alpha\{\cdot|\xi(0) = 1\}$, where ν_α is the homogeneous product measure on $\{0,1\}^{\mathbb{Z}^d}$ with constant density $0 < \alpha < 1$. We define a modified exclusion model on the lattice $\mathbb{Z}$ as follows: the tagged particle jumps to neighbours at distance $\pm 1, 2$ with rate $\frac{1}{4}$ and the other particles perform nearest neighbour jumps with rate $\frac{1}{2}$. Our main result is the following:

Theorem 1. The position of the tagged particle, X_t, satisfies $X_t/\sqrt{t}$ converges in distribution to a normal random variable with non-zero variance and zero mean.

For a general (not necessarily symmetric) exclusion process on $\mathbb{Z}^d$, let $\overline{\Omega}$ be the generator of the tagged particle process and $D(u)$ the Dirichlet form of a (sufficiently nice) function u on $\{\xi : \xi(0) = 1\}$, i.e. $D(u) = D_{sh}(u) + D_{ex}(u)$, where $D_{sh}(u)$ and $D_{ex}(u)$ are given by

$$D_{sh}(u) = \frac{1}{2}\int \sum_{x\in\mathbb{Z}^d\setminus\{0\}} p(0,x)(u(\tau_x\xi) - u(\xi))^2(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)$$

$$D_{ex}(u) = \frac{1}{4}\int \sum_{x,y\in\mathbb{Z}^d\setminus\{0\}} p(x,y)(u(\xi_{x,y}) - u(\xi))^2\, d\overline{\nu_\alpha}(\xi).$$

We let ψ denote the drift of the tagged particle given by $\psi(\xi) = \sum xp(0,x)(1-\xi(x))$, $\overline{\psi}$ the centered drift given by $\overline{\psi} = \psi - \int \psi(\xi)\, d\overline{\nu_\alpha}(\xi)$ and u_λ for $\lambda > 0$ is defined via $\lambda u_\lambda - \overline{\Omega} u_\lambda = \overline{\psi}$. If condition H_{-1} holds, i.e. there exists a positive constant C such that both inequalities

$$\left|\int \overline{\psi}(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \leq C\sqrt{D(u)} \tag{1}$$

and

$$\left|\int u(\xi)(\overline{\Omega} u_\lambda)(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \leq C\sqrt{D(u)} \tag{2}$$

hold for all $\lambda > 0$ and all (sufficiently nice) functions u then Liggett proved that $\frac{X_t - \mathbb{E}X_t}{\sqrt{t}}$ converges in distribution to a mean zero Gaussian random variable (Theorem 4.50 on page 295 of [21]). The condition was shown to hold when $p(\cdot,\cdot)$ has mean zero (i.e. $\sum xp(0,x) = 0$) by Varadhan in 1995 ([33]) and for non-zero mean for $d \geq 3$ by Sethuraman, Varadhan and Yau in 2000 ([29]). The Gaussian random variable might in fact be degenerate. This was proven by Arratia in 1983 ([3]) for nearest neighbour symmetric jumps for $d = 1$. In fact he proved that $\frac{X_t}{t^{0.25}}$ converges in distribution to a Gaussian random variable with variance $\sqrt{2/\pi}(1-\alpha)/\alpha$. Non-degeneracy was proven for nearest-neighbour non-symmetric transition rates on $\mathbb{Z}$ by Kipnis in 1986 ([14]) and in the aforementioned cases for which conditions (1) and (2) hold that do not fall under Arratia's treatment. Landim et al. ([19]) also studied an exclusion process model on $\mathbb{Z}$ in which the tagged particle behaves differently from all the other particles. In their model the tagged particle performs asymmetric nearest neighbour jumps while the rest of the particles perform symmetric nearest neighbour jumps. However, I'm not aware of any analogue of Theorem 1 in which the tagged particle behaves differently from the other particles.

The exclusion process has many applications to other areas of science. TASEP, totally asymmetric exclusion process, was first introduced in 1968 to describe ribosome motion along a piece of mRNA during translation ([25]). In its simplest form, the model consists of a one-dimensional lattice of N points, denoted by $i = 1, \cdots, N$, and with spacing $a = L/N$, where L is the total length of the lattice (typically, we set $L = 1$). The most common boundary conditions are as follows: particles are added to the left boundary of the lattice ($i = 1$) at rate α and removed from the right boundary of the lattice ($i = N$) at rate β. Particles on the lattice attempt jumps to their right neighbouring site at rate $p = 1$, provided that the destination sites are unoccupied. This toy model serves as a description of a ribosome moving from codon to codon on an mRNA strand. We can think of each particle on the lattice as a ribosome. Just like the particle, the ribosome attaches to the mRNA at the start codon (the left boundary of the lattice). Then the ribosome moves along the mRNA strand in a specific direction called the $5'$ to $3'$ direction, translating one codon in the mRNA at a time. This naturally corresponds to the asymmetric nature of TASEP. In addition, TASEP captures the most basic ribosome-ribosome property by forbidding two ribosomes from occupying the same codon.

Since its introduction, the original TASEP model has been modified to transform it into a more realistic model. One important modification, called TASEP/LK, where LK stands for Langmuir kinetics, was introduced by Parmeggiani et al. ([28]). TASEP/LK is defined as follows: TASEP is extended with the possibility of particles attaching to the lattice at rates ω_A and detaching from a lattice and moving to an either infinite or finite reservoir at rate ω_D. TASEP/LK is useful for studying molecular motors, since they are able to attach and detach from their associated filaments or “tracks”. In fact, there is a high variability of the rates at which attachment and detachment occurs, and the variability is related to the biological function of the motor ([1],[32]).

Another modification that can be added to TASEP is the use of multiple coupled one-dimensional lattices. In particular, we can couple two TASEP “lanes” together by letting particles hop back and forth between the lanes with some characteristic rates s_1 and s_2. Such a process is relevant to studying molecular motors which move along a set of parallel tracks ([26],[11]). TASEP coupled with multi-lane SEPs (symmetric exclusion processes) is useful for modelling vehicular traffic ([34]).

Another natural modification of TASEP which models the movement of ants, called the unidirectional ant-trail model (ATM), was introduced by Chowdhurry et al. in 2002 ([5]). In ATM ants move strictly on a one-dimensional lattice with L sites. Each site can either be occupied by one of N ants or be unoccupied. Ants leave marks on sites which they occupy called pheromones, so each site is either marked or unmarked by a pheromone. If a site is not occupied by an ant but contains a pheromone mark, then the pheromone mark evaporates at rate f. Otherwise, whenever an ant occupies a site then the site also contains a pheromone mark and the mark only starts evaporating once the ant leaves the site. Unlike in TASEP, the hopping rate of particles p at site i is not constant and depends on the existance of pheromone marks at site $i + 1$. If site $i + 1$ is unoccupied by an ant but contains a pheromone mark, then the hopping rate is $p = Q$, while if the site $i + 1$ is unoccupied by an ant and does not contain a pheromone mark, then $p = q < Q$. So the presence of pheromone marks leads to an increase in the hopping rate from q to Q. Chowdhurry et al. pointed out the relations between the unidirectional ant-trail model and the bus route model (BRM) and the vehicular traffic on freeways model called the Nagel-Schreckenberg model (NS model). Chowdhurry et al. also pointed out in 2004 the relationship to the zero range process (ZRP) ([18]). In addition, the density fluctuation field associated to the accelerated generator of asymmetric exclusion converges in law in Skorohod space to the stationary energy solution of the Burgers equation (for more details see [10]).

The exclusion process is similar to the random stirring process (also known as the random interchange process), a random process on graphs where a different particle is placed on each of the vertices and Poisson clocks are placed on the edges and whenever a clock on an edge rings the particles at the endpoints of the edge switch positions (we replaced clocks on vertices with clocks on edges, all the particles are different, and once the clock rings the step is deterministic and does not depend on the weight given to the neighbours). We can think of it as a sequence of random transpositions applied to the identity permutation. The existence of large cycles in the random permutation after some time has been studied on trees ([13]), the complete graph on n elements (see e.g. [4]) and the hypercube ([17]). Note in the case of the complete graph, the stirring process is obtained by simply applying a sequence of random transpositions. The stirring process on the complete graph can be thought of as the simplest card shuffling method, whereby at each shuffling step two random cards are removed from the deck of cards and exchanged. This simple card shuffling technique was analyzed by Diaconis and Shahshahani ([7]). Other card shuffling methods were also explored, including the riffle shuffle (the deck is roughly divided into half and the two halves are interleaved) ([2]), the cyclic-to-random shuffle (at step t the random card selected is exchanged with the card at position t mod n) ([31]) and the "semi-random transposition" shuffle (any shuffle in which a random card is exchaged with another card chosen according to an arbitrary rule which is either deterministic or random) ([27]).

2. Preliminaries

The results and definitions in this section can be found in chapter I of Liggett's book Interacting Particle Systems ([23]) and in chapter 3 of Liggett's other book Continuous Time Markov Processes ([24]). Let X be either a compact or a locally compact metric space with measurable structure given by the σ-algebra of Borel sets $\mathcal{B}$. We say a real-valued function f on a locally compact space X vanishes at infinity if for each $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. Let $C(X)$ denote the collection of real-valued continuous functions on X in the compact case or the collection of real-valued continuous functions on X vanishing at infinity in the locally compact case and in both cases the space is equipped with the supremum norm. Let $\mathcal{P}$ denote the set of probability measures on X endowed with the topology of weak convergence: $\mu_n \to \mu$ iff $\int f\, d\mu_n \to \int f\, d\mu$ for all $f \in C(X)$.

Let $D[0,\infty)$ denote the set of right continuous functions $\eta_\cdot : [0,\infty) \to X$ with left limits. For $s \in [0,\infty)$ let $\pi_s : D[0,\infty) \to X$ be defined via $\pi(\eta_\cdot) = \eta_s$. Let $\mathcal{F}$ be the smallest σ-algebra on $D[0,\infty)$ relative to which all the mappings π_s are measurable. For $t \in [0,\infty)$ let $\mathcal{F}_t$ be the smallest σ-algebra on $D[0,\infty)$ relative to which all the mappings π_s for $s \leq t$ are measurable.

Definition 1. A *Markov process* on X is a collection $\{\mathbb{P}^\eta, \eta \in X\}$ of probability measures on $D[0,\infty)$ indexed by X with the following properties:

(i) $\mathbb{P}^\eta\big[\{\xi_\cdot \in D[0,\infty) : \xi_0 = \eta\}\big] = 1$ for all $\eta \in X$.

(ii) The mapping $\eta \to \mathbb{P}^\eta(A)$ from X to $[0,1]$ is measurable for each $A \in \mathcal{F}$.

(iii) $\mathbb{P}^\eta\left[\eta_{s+\cdot} \in A \middle| \mathcal{F}_s\right] = \mathbb{P}^{\eta_s}(A)$ $(\mathbb{P}^\eta)$-a.s. for each $\eta \in X$ and $A \in \mathcal{F}$ and each $s \geq 0$.

The expectation corresponding to $\mathbb{P}^\eta$ will be denoted by $\mathbb{E}^\eta$. Thus, $\mathbb{E}^\eta(Z) = \int_{D[0,\infty)} Z\, d\mathbb{P}^\eta$ for any measurable function Z on $D[0,\infty)$ which is integrable relative to $\mathbb{P}^\eta$. For $f \in C(X)$ we write $(S(t)f)(\eta) = \mathbb{E}^\eta f(\eta_t)$.

Definition 2. A Markov process $\{\mathbb{P}^\eta, \eta \in X\}$ is said to be a *Feller process* if $S(t)f \in C(X)$ for each $t \geq 0$ and $f \in C(X)$.

We recall the definition of a Markov pregenerator.

Definition 3. A (usually unbounded) linear operator Ω on $C(X)$ with domain $\mathcal{D}(\Omega)$ is a *Markov pregenerator* if it satisfies

(i) If X is compact: $1 \in \mathcal{D}(\Omega)$ and $\Omega 1 = 0$. In the non-compact case: for small positive λ there exists $f_n \in \mathcal{D}(\Omega)$ so that $g_n = f_n - \lambda\Omega f_n$ satisfies $\sup_n ||g_n||_\infty < \infty$ and both f_n and g_n converge to 1 pointwise.

(ii) $\mathcal{D}(\Omega)$ is dense in $C(X)$.

(iii) If $f \in \mathcal{D}(\Omega)$, $\lambda \geq 0$ and $f - \lambda\Omega f = g$, then

$$\inf_{\xi \in X} f(\xi) \geq \inf_{\xi \in X} g(\xi). \tag{3}$$

In order to verify condition (iii) the following criterion is useful in the compact case, which appears in Liggett ([23]) as Proposition 2.2.

Lemma 1. Suppose that the linear operator Ω on $C(X)$ satisfies the following property: if $f \in \mathcal{D}(\Omega)$ and η is such that $f(\eta) = \min_{\xi \in X} f(\xi)$, then $(\Omega f)(\eta) \geq 0$. Then Ω satisfies condition (iii) of Definition 3.

We need the following definition for Lemma 2.

Definition 4. A linear operator Ω on $C(X)$ is said to be *closed* if its graph is a closed subset of $C(X) \times C(X)$. A linear operator $\overline{\Omega}$ is called *the closure* of Ω if $\overline{\Omega}$ is the smallest closed extension of Ω.

Lemma 2. Suppose Ω is a Markov pregenerator. Then Ω has a closure $\overline{\Omega}$ which is also a Markov pregenerator.

We are now ready to define a Markov generator.

Definition 5. A closed Markov pregenerator Ω is called a *Markov generator* if the range of $I - \lambda\Omega$ satisfies

$$\mathcal{R}(I - \lambda\Omega) = C(X) \tag{4}$$

for all sufficiently small $\lambda \geq 0$.

We note that a Markov generator satisfies the following stronger property, which is Proposition 2.8 in chapter I of Liggett ([23].)

Lemma 3. (i) A bounded Markov pregenerator is a Markov generator.

(ii) A Markov generator satisfies $\mathcal{R}(I - \lambda\Omega) = C(X)$ for all $\lambda \geq 0$.

We recall the definition of a Markov semigroup.

Definition 6. A family $\{S(t), t \geq 0\}$ of continuous linear operators on $C(X)$ is called a *Markov semigroup* if it satisfies

(i) $S(0) = I$, the identity operator on $C(X)$.

(ii) The mapping $t \to S(t)f$ from $[0, \infty)$ to $C(X)$ is right continuous for each $f \in C(X)$.

(iii) $S(t+s)f = S(t)S(s)f$ for each $f \in C(X)$ and all $s, t \geq 0$.

(iv) If X is compact: $S(t)1 = 1$ for all $t \geq 0$. In the non-compact case: there exist $f_n \in C(X)$ so that $\sup_n ||f_n||_\infty < \infty$ and $S(t)f_n$ converges to 1 pointwise for each $t \geq 0$.

(v) $S(t)f \geq 0$ for all $t \geq 0$ whenever $f \in C(X)$ is non-negative.

We note that in the compact case if we apply $S(t)$ to $||f|| \pm f$, then by (iv) and (v) we conclude that $S(t)$ is a contraction semigroup, i.e. satisfies $||S(t)f||_\infty \leq ||f||_\infty$ for all $f \in C(X)$ and all $t \geq 0$. The relation between Markov semigroups and Feller processes is given via Theorem 3.26 in chapter 3 of [24].

Lemma 4. If $S(t)$ is a Markov semigroup, then there is a Feller process $\{\mathbb{P}^\eta, \eta \in X\}$ satisfying

$$\mathbb{E}^\eta f(\eta_t) = (S(t)f)(\eta) \tag{5}$$

for all $\eta \in X$,$t \geq 0$ and $f \in C(X)$.

The distribution of the stochastic process (η_t) is given in the definition below.

Definition 7. Suppose $\{S(t), t \geq 0\}$ is a Markov semigroup on $C(X)$. Given $\mu \in \mathcal{P}$, $\mu S(t) \in \mathcal{P}$ is defined via the relation

$$\int f\, d[\mu S(t)] = \int S(t) f\, d\mu \tag{6}$$

for each $f \in C(X)$. The probability measure $\mu S(t)$ is interpreted as the distribution of η_t when η is distributed according to μ.

Definition 8. We say that $\mu \in \mathcal{P}$ is *stationary* for the process (η_t) if $\mu S(t) = \mu$ for all $t \geq 0$. We let $\mathcal{I}$ denote the class of stationary measures for the process (η_t) and let $\mathcal{I}_e$ denote its extreme points.

The stationary measures of the process are determined by the generator via the following result, which is part of Proposition 1.8 in chapter I in Liggett ([23]):

Theorem 2. A probability measure μ on $\{0, 1\}^S$ is stationary for (η_t) iff

$$\int \Omega f\, d\mu = 0$$

for all cylinder functions f.

Proposition 1.8 from Liggett also provides the following result in the compact case:

Theorem 3. $\mathcal{I}$ is a non-empty compact convex set.

The relation between Markov generators and Markov semigroups is given via the following theorem, which can be found in chapter IX of Yosida ([35]).

Theorem 4. (Hille-Yosida) There is a one-to-one correspondence between Markov generators Ω on $C(X)$ and Markov semigroups $S(t)$ on $C(X)$. The correspondence is given by

(i) $\mathcal{D}(\Omega) = \{f \in C(X) : \lim_{t \downarrow 0} \frac{S(t)f - f}{t}$ exists$\}$ and $\Omega f = \lim_{t \downarrow 0} \frac{S(t)f - f}{t}$ for all $f \in \mathcal{D}(\Omega)$.

(ii) $S(t)f = \lim_{n \to \infty} (I - \frac{t}{n}\Omega)^{-n} f$ for all $f \in C(X)$ and $t \geq 0$.

In addition, if $f \in \mathcal{D}(\Omega)$ then $S(t)f \in \mathcal{D}(\Omega)$ for all $t \geq 0$ and for all $s > 0$, $(d/ds)S(s)f = \Omega S(s) f = S(s)\Omega f$.

Note that Ω can only be defined on $\mathcal{D}(\Omega)$, a dense subset of $C(X)$ in part (i), while by Def. 5 $(I - \lambda\Omega)^{-1}$ is defined on all $C(X)$ for all $\lambda > 0$ sufficiently small, so part (ii) can in fact be defined on $C(X)$.

Ω is called the generator of $S(t)$ and $S(t)$ is the semigroup generated by Ω. Given an initial distribution μ we say that the stochastic process (η_t) whose distribution at time t is given by $\mu S(t)$ is governed by Ω. We define stationarity and ergodicity of the process (η_t).

Definition 9. A stochastic process (η_t) on X is said to be *stationary* if the joint distributions of

$$(\eta_{t_1+t}, ..., \eta_{t_n+t})$$

are independent of t for all choices of n and $t_1, ..., t_n$.

Definition 10. We call a stationary stochastic process (η_t) *ergodic* if $\mathbb{P}(G) \in \{0,1\}$ for every event G in path space which is invariant under time shifts, i.e. satisfies $\eta_\cdot \in G \Rightarrow \eta_{s+\cdot} \in G$ for all $s > 0$.

The following result holds in the compact case. It appaears as theorem B52 in the section "Background and Tools" of [21].

Theorem 5. Suppose that (η_t) is a stationary Markov process whose distribution at each fixed time is the measure $\mu \in \mathcal{I}$. Then each of the following is equivalent to the ergodicity of the process.

(i) $\mu \in \mathcal{I}_e$.

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}F(\eta_0)G(\eta_s)\,ds = \int F\,d\mu \int G\,d\mu$ for all bounded continuous functions F, G.

3. Interacting Particle systems

The processes we discuss are stochastic processes (η_t) on the compact configuration space $\{0,1\}^S$, where S is a countable set. The infinitesimal generator governing (η_t), which we denote by $\overline{\Omega}$, is the closure in $C(\{0,1\}^S)$ of the operator Ω which takes the following form when applied to cylinder functions (i.e. functions depending on a finite set of coordinates)

$$\Omega f(\eta) = \sum_{\zeta} c(\eta,\zeta)[f(\zeta) - f(\eta)] \tag{7}$$

where the transition rates from η to ζ, $c(\eta,\zeta)$, are chosen such that $\overline{\Omega}$ satisfies $\mathcal{R}(I - \lambda\overline{\Omega}) = C(\{0,1\}^S)$ for sufficiently small $\lambda \geq 0$. We remark that $\overline{\Omega}$ is clearly a closed Markov pregenerator as the closure of the Markov pregenerator Ω by Lemma 2 - the fact that Ω is a Markov pregenerator follows from the Stone-Weierstrass theorem and Lemma 1.

4. Examples

We define the important models in the field. The following notation is needed in order to describe the transitions. For a configuration $\eta \in \{0,1\}^S$ we define for each $x, y \in S$ the functions $\eta_x, \eta_{x,y} \in \{0,1\}^S$ as

$$\eta_x(a) = \begin{cases} 1 - \eta(a) & \text{if } a = x \\ \eta(a) & \text{if } a \neq x \end{cases} \tag{8}$$

and

$$\eta_{x,y}(a) = \begin{cases} \eta(y) & \text{if } a = x \\ \eta(x) & \text{if } a = y \\ \eta(a) & \text{if } a \neq x, y \end{cases} \tag{9}$$

We note that all the infinitesimal generators in this section are well defined by Section III in chapter I in Liggett ([23]).

4.1. **The contact process.**

The contact process η_t with contact rate $\lambda > 0$ and recovery rate 1 on a bounded degree graph S is governed by the generator $\overline{\Omega}$ which is the closure in $C(\{0,1\}^S)$ of the operator Ω which takes the following form when applied to cylinder functions

$$\Omega f(\eta) = \sum_{x\in S} [\eta(x) + \lambda s(\eta,x)(1-\eta(x))][f(\eta_x) - f(\eta)] \tag{10}$$

where $s(\eta, x) = \sum_{y \sim x} \eta(y)$ (we use the notation $y \sim x$ to mean that y and x are connected by an edge). We interpret sites with $\eta(x) = 1$ as infected and sites with $\eta(x) = 0$ as healthy. Let δ_0 denote the pointmass on the configuration $\eta \equiv 0$ which by Theorem 2 is stationary. The next theorem can be found in Part I of Liggett ([21]).

Theorem 6. For $S = \mathbb{Z}^d$, the d-dimensional integer lattice, there exists a critical value $\lambda(d) \in (\frac{1}{2d-1}, \frac{2}{d})$ so that

(i) $\lambda \leq \lambda(d)$ implies that $\mathcal{I} = \{\delta_0\}$ and $\eta_t \to \delta_0$ weakly as $t \to \infty$ for any initial configuration η_0 and

(ii) $\lambda > \lambda(d)$ implies that $\mathcal{I}_e = \{\delta_0, \nu\}$ for some $\nu \neq \delta_0$ and $\eta_t \to \nu$ weakly for any initial configuration η_0 with infinitely many infected sites.

Note that for $S = \mathbb{Z}$, $\lambda = \lambda(1)$ and $\eta_0 \equiv 1$ by Theorem 6, $\eta_t \to \delta_0$ weakly. However we also have the following result, which appears in Theorem 3.10 of chapter VI of Liggett ([23]):

Theorem 7. For $S = \mathbb{Z}$, $\lambda = \lambda(1)$ and $\eta_0 \equiv 1$ the following holds:

(i) For each x, $\lim_{t \to \infty} t\mathbb{P}^{\eta_0}(\eta_t(x) = 1) = \infty$ and furthermore

(ii) $\mathbb{P}^{\eta_0}(\forall s > 0\ \exists t \geq s \text{ such that } \eta_t(x) = 1) = 1$.

The next theorem can be found in Part I of Liggett ([21]).

Theorem 8. For $S = \mathrm{T}_d$ ($d \geq 2$), the tree in which every vertex has $d+1$ neighbours, there are two critical values $\lambda_1(d) < \lambda_2(d)$, $\frac{1}{d+1} \leq \lambda_1(d) \leq \frac{1}{d-1}$ and $\frac{1}{2\sqrt{d}} \leq \lambda_2(d) \leq \frac{1}{\sqrt{d}-1}$ so that

(i) $\lambda \leq \lambda_1(d)$ implies that $\mathcal{I} = \{\delta_0\}$ and $\eta_t \to \delta_0$ weakly for any initial configuration η_0 and

(ii) $\lambda_1(d) < \lambda \leq \lambda_2(d)$ implies that $\mathcal{I}_e$ is infinite and

(iii) $\lambda > \lambda_2(d)$ implies that $\mathcal{I}_e = \{\delta_0, \nu\}$ for some $\nu \neq \delta_0$ and $\eta_t \to \nu$ weakly for any initial configuration η_0 with infinitely many infected sites.

In case (ii), if the initial configuration η_0 has finitely many infected sites then

$$\mathbb{P}^{\eta_0}(\eta_t \not\equiv 0\ \forall t > 0) > 0$$

but $\forall x \in S$

$$\mathbb{P}^{\eta_0}(\exists N \text{ so that } \eta_t(x) = 0\ \forall t \geq N) = 1.$$

4.2. **The linear voter model.** Here S is an arbitrary countable set and $p(x, y)$ for $x, y \in S$ satisfy $p(x, y) \geq 0$ and $\sum_{y \in S} p(x, y) = 1$ for all $x \in S$. The generator $\overline{\Omega}$ of the linear model process η_t is the closure in $C(\{0, 1\}^S)$ of Ω which when applied to cylinder functions takes the form

$$\Omega f(\eta) = \sum_{x \in S} s(\eta, x)[f(\eta_x) - f(\eta)] \tag{11}$$

with $s(\eta, x) = \sum_{y \in S} p(x, y)|\eta(y) - \eta(x)|$. The interpretation is that each site has two opinions, 0 and 1, and changes its opinion at a rate which is the weighted average of its neighbours which have a differing opinion. An alternative interpretation is in terms of spatial conflict. Two nations control the areas $\{x : \eta(x) = 0\}$ and $\{x : \eta(x) = 1\}$ respectively and a change of value at a point x represents an expansion of one nation's area at the expense of the other nation. The trivial stationary distributions (satisfying Theorem 2) are the the pointmasses

δ_0 and δ_1 on $\eta \equiv 0$ and $\eta \equiv 1$ respectively. Let $p^{(n)}(x,y)$ be the n-step transition probabilities associated with $p(x,y)$ defined recursively via the equations

$$\begin{aligned} p^{(0)}(x,x) &= 1 \\ p^{(1)}(x,y) &= p(x,y) \\ p^{(n)}(x,y) &= \sum_{z \in S} p^{(n-1)}(x,z)p(z,y). \end{aligned}$$

Now let $X(t)$ and $Y(t)$ be independent copies of the continuous time Markov chain (i.e. continuous time random walks on S) with transition probabilities

$$p_t(x,y) = e^{-t} \sum_{n=1}^{\infty} \frac{t^n}{n!} p^{(n)}(x,y)$$

and let $Z(t) = X(t) - Y(t)$. We restrict ourselves to the case $S = \mathbb{Z}^d$ and $p(x,y) = p(0, y-x)$ for each $x, y \in S$. In addition we assume that the Markov chain on S with transition rates $p(x,y)$ is irreducible (i.e. for each $x \neq y \in S$ there exists a $k(x,y) > 0$ such that $p^{(k(x,y))}(x,y) > 0$). If $Z(t)$ is recurrent then we have the following result, which appears as Theorem 3 on page 22 in Liggett's lectures ([22]).

Theorem 9. (i) For each $\eta \in \{0,1\}^S$ and every $x, y \in S$,

$$\lim_{t \to \infty} P^{\eta}(\eta_t(x) \neq \eta_t(y)) = 0.$$

(ii) $\mathcal{I}_e = \{\delta_0, \delta_1\}$.

(iii) If μ satisfies $\mu\{\eta : \eta(x) = 1\} = \lambda$ for all $x \in S$ then $\mu S(t)$ converges weakly to $\lambda\delta_1 + (1-\lambda)\delta_0$.

For $0 \le \alpha \le 1$ define ν_α to be the homogeneous product measure on $\{0,1\}^S$ with density α, i.e.

$$\nu_\alpha\{\eta : \eta \equiv 1 \text{ on } A\} = \alpha^{|A|} \tag{12}$$

for each finite set $A \subset S$.

Definition 11. We call a set $A \subseteq \{0,1\}^{\mathbb{Z}^d}$ *shift invariant* if for all $z \in \mathbb{Z}^d$ and for all $(\eta(i))_{i \in \mathbb{Z}^d} \in A \Rightarrow (\eta(i+z))_{i \in \mathbb{Z}^d} \in A$. We define $A + z = \{(\eta(i+z))_{i \in \mathbb{Z}^d} : (\eta(i))_{i \in \mathbb{Z}^d} \in A\}$.

Definition 12. We call a probability measure *translation invariant* if, for any event A and any $a \in \mathbb{Z}^d$, it assigns the same probability to A and $A + a$.

Definition 13. A translation invariant probability measure on $\{0,1\}^{\mathbb{Z}^d}$ is said to be *spatially ergodic* if it assigns probability 0 or 1 to every shift invariant subset of $\{0,1\}^{\mathbb{Z}^d}$.

If $Z(t)$ is transient then the following result appears in Theorem 5 on page 25 in Liggett's lectures ([22]).

Theorem 10. Fix $0 \le \alpha \le 1$ and let $\eta_0 \sim \nu_\alpha$ then

(i) $\nu_\alpha S(t)$ converges weakly to μ_α as $t \to \infty$.

(ii) μ_α is translation invariant and spatially ergodic.

(iii) $\mu_\alpha\{\eta : \eta(x) = 1\} = \alpha$ for each $x \in S$ and

(iv) $Cov_{\mu_\alpha}[\eta(x), \eta(y)] = \alpha(1-\alpha)P^{Z(0)=x-y}(Z(t) = 0 \text{ for some } t \ge 0)$.

In addition we also have this result in the transient case, which is Theorem 6 on page 26 in Liggett's lectures ([22]):

Theorem 11. (i) $\mathcal{I}$ is the closed convex hull of $\{\mu_\alpha : 0 \le \alpha \le 1\}$.

(ii) $\mathcal{I}_e = \{\mu_\alpha : 0 \le \alpha \le 1\}$.

4.3. **The exclusion process.** Here S is an arbitrary countable set and the transition rates $p(x, y)$ for $x, y \in S$ satisfy $p(x, y) \ge 0$, $p(x, x) = 0$ and $\sum_{y \in S} p(x, y) = 1$ for all $x \in S$ and $\sup_{y \in S} \sum_{x \in S} p(x, y) < \infty$. The generator $\overline{\Omega}$ of the exclusion process η_t is the closure in $C(\{0,1\}^S)$ of the operator Ω which when applied to cylinder functions takes the form

$$\Omega f(\eta) = \sum_{x \in S} p(x, y)\eta(x)(1 - \eta(y))[f(\eta_{x,y}) - f(\eta)]. \tag{13}$$

The invariant measures for the exclusion process are closely related to the bounded harmonic functions for $p(x, y)$. Let

$$\mathcal{H} = \left\{\alpha : S \to [0,1] : \sum_{y \in S} p(x, y)\alpha(y) = \alpha(x)\ \forall x \in S\right\} \tag{14}$$

denote the set of harmonic functions with respect to $p(x, y)$ taking values between 0 and 1 and let $p_t(x, y)$ be as defined in Section 4.2. For $\alpha \in \mathcal{H}$ let ν_α be the product measure on $\{0,1\}^S$ with marginals given by

$$\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x). \tag{15}$$

The following result, which appears as Theorem 1 on page 40 of Liggett's lectures ([22]) shows the product measures are stationary in some simple instances.

Theorem 12. (i) If $p(\cdot, \cdot)$ is doubly stochastic, i.e.

$$\sum_{x \in S} p(x, y) = 1\ \forall y \in S$$

then $\nu_\alpha \in \mathcal{I}$ for all constants $0 \le \alpha \le 1$.

(ii) If π is a non-negative function on S and $p(\cdot, \cdot)$ is reversible with respect to π, i.e.

$$\pi(x)p(x, y) = \pi(y)p(y, x)\ \forall x, y \in S$$

then $\nu_\alpha \in \mathcal{I}$ where

$$\alpha(x) = \frac{\pi(x)}{1 + \pi(x)}\ \forall x \in S.$$

The proofs of the next two results can be found in chapter VIII of Liggett ([23]). The next result holds under the assumption that $S = \mathbb{Z}^d$ and the Markov chain on S with transition rates $p(x, y)$ is irreducible and the rates satisfy $p(x, y) = p(0, y - x) = p(y, x)$ for each $x, y \in S$.

Theorem 13. (i) $\mathcal{I}_e = \{\nu_\alpha$: constant $0 \le \alpha \le 1\}$

(ii) If μ is translation invariant and spatially ergodic, then $\mu S(t)$ converges weakly to ν_ρ as $t \to \infty$ where $\rho = \mu\{\eta : \eta(0) = 1\}$.

The following result holds if the Markov chain on S with transition probabilities $p(x, y)$ is irreducible and symmetric (i.e. $p(x, y) = p(y, x)$ for all $x, y \in S$).

Theorem 14. (i) For each $\alpha \in \mathcal{H}$, $\nu_\alpha S(t)$ converges weakly to a measure μ_α.

(ii) $\mu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x)$ for all $x \in S$ and

$$\mu_\alpha\{\eta : \eta(x) = 1, \eta(y) = 1\} \leq \alpha(x)\alpha(y)$$

for all $x \neq y \in S$.

(iii) μ_α is a product measure if and only if α is a constant.

(iv) $\mathcal{I}_e = \{\mu_\alpha : \alpha \in \mathcal{H}\}$.

(v) If the probability measure μ on $\{0,1\}^S$ satisfies

$$\lim_{t\to\infty} \sum_{y \in S} p_t(x,y)\mu\{\eta : \eta(y) = 1\} = \alpha(x)$$

for every $x \in S$ and

$$\lim_{t\to\infty} \sum_{y_1, y_2 \in S} p_t(x_1, y_1)p_t(x_2, y_2)\mu\{\eta : \eta(y_1) = 1, \eta(y_2) = 1\} = \alpha(x_1)\alpha(x_2)$$

for every $x_1, x_2 \in S$ then $\alpha \in \mathcal{H}$ and $\mu S(t)$ converges weakly to μ_α as $t \to \infty$.

4.3.1. *The tagged particle process on $\mathbb{Z}^d$.* Let X_t denote the position of a tagged particle at time t and let η_t be the process governed by the exclusion process. Let $\xi_t(x) = \eta_t(X_t + x)$ be the process viewed from the particle. For a configuration $\xi \in \{0,1\}^{\mathbb{Z}^d} \cap \{\xi : \xi(0) = 1\}$ we define the spacial shift via

$$\tau_x \xi(a) = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a = -x \\ \xi(a + x) & \text{if } a \neq -x, 0 \end{cases} \tag{16}$$

We assume the transition rates $p(x,y)$ are translation invariant and finite range (i.e. $p(x,y) = 0$ whenever the distance between x and y is larger than some positive constant) and the Markov chain on Z^d with transition rates $p(x,y)$ is irreducible. The tagged particle process ξ_t is governed by the generator $\overline{\Omega}$ which is the closure in $C(\{0,1\}^S \cap \{\xi : \xi(0) = 1\})$ of Ω which takes the following form when applied to cylinder functions

$$\Omega f(\xi) = \Omega_{ex} f(\xi) + \Omega_{sh} f(\xi), \tag{17}$$

where Ω_{ex} and Ω_{sh} are given by

$$\Omega_{ex} f(\xi) = \sum_{x,y \in \mathbb{Z}^d \setminus \{0\}} p(x,y)\xi(x)(1 - \xi(y))[f(\xi_{x,y}) - f(\xi)] \tag{18}$$

$$\Omega_{sh} f(\xi) = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} p(0,x)(1 - \xi(x))[f(\tau_x \xi) - f(\xi)] \tag{19}$$

The notation sh and ex corresponds to shifts and exchanges respectively. We note that Landim et al. and Ferrari studied the tagged particle process on $\mathbb{Z}$ ([19],[9]). In Landim's model all the untagged particles perform nearest neighbour symmetric jumps while the tagged particle performs asymmetric nearest neighbour jumps. Ferrari studied the tagged particle process for translation invariant asymmetric nearest neighbour jumps. For each probability measure ν on $\{0,1\}^S$ let $\overline{\nu} = \nu\{\cdot|\xi(0) = 1\}$. The following result was proven by Liggett ([21]) for the non-nearest neighbour case in $\mathbb{Z}$. We also prove it for our model, in which $S = \mathbb{Z}$ and $p(0,1) = p(0,2) = p(0,-1) = p(0,-2) = 1/4$ and $p(x, x+1) = 1/2$ for $x \neq 0, -1$ and $p(x, x-1) = 1/2$ for $x \neq 0, 1$.

Theorem 15. For each constant $0 < \alpha < 1$, The process (ξ_t) started from $\overline{\nu_\alpha}$ is stationary and ergodic in the sense of Def. 10 for the tagged particle process.

For the nearest neighbour symmetric case in $\mathbb{Z}$ (i.e. $S = \mathbb{Z}$ and $p(x, x-1) = p(x, x+1) = 1/2$ for each $x \in S$) the following result was proven by Arratia ([3]).

Theorem 16. For each initial distribution $\overline{\nu_\alpha}$ $(0 < \alpha < 1)$, $\frac{X_t}{t^{0.25}}$ converges in distribution to a Gaussian random variable with variance $\sqrt{2/\pi}(1-\alpha)/\alpha$.

The following result was proven for a number of different settings, which include for Z the symmetric non-nearest neighbour case ([15],[21]) and the non-symmetric nearest neighbour case on $\mathbb{Z}$ ([14]). The result also holds for our model.

Theorem 17. For each initial distribution $\overline{\nu_\alpha}$ $(0 < \alpha < 1)$, $\frac{X_t - \mathbb{E}(X_t)}{\sqrt{t}}$ converges in distribution to a zero mean Gaussian random variable with non-zero variance.

As pointed out by Liggett ([21]), Theorem 17 follows from the following technical condition, which does not hold in the one dimensional nearest-neighbour case. Before we state the condition we need a few definitions. We define the drift $\psi(\xi) = \sum_{x \in S} p(0, x)x(1 - \xi(x))$ for each $\xi \in \{0, 1\}^S \cap \{\xi : \xi(0) = 1\}$ and the centered drift via $\overline{\psi} = \psi - \int \psi(\xi)\, d\overline{\nu_\alpha}(\xi)$. In addition, we define the Dirichlet form for each measurable function a on $\{0, 1\}^{\mathbb{Z}^d} \cap \{\xi : \xi(0) = 1\}$ as $D(a) = D_{ex}(a) + D_{sh}(a)$, where

$$D_{sh}(a) = \frac{1}{2} \int \sum_{x \in \mathbb{Z}^d \setminus \{0\}} p(0, x)(a(\tau_x \xi) - a(\xi))^2 (1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)$$

$$\begin{aligned} D_{ex}(a) &= \frac{1}{4} \int \sum_{x,y \in \mathbb{Z}^d \setminus \{0\}} p(x, y)(u(\xi_{x,y}) - u(\xi))^2\, d\overline{\nu_\alpha}(\xi) \\ &= \frac{1}{2} \int \sum_{x,y \in \mathbb{Z}^d \setminus \{0\}} p(x, y)(a(\xi_{x,y}) - a(\xi))^2 \xi(x)(1 - \xi(y))\, d\overline{\nu_\alpha}(\xi). \end{aligned}$$

The technical condition is as follows:

$$\left| \int \psi(\xi) u(\xi)\, d\overline{\nu_\alpha} \right| \le C \sqrt{D_{ex}(u)} \tag{20}$$

holds for all cylinder functions u for some positive $C > 0$. This condition is a tightening of part of condition H_{-1} which we define below.

Definition 14. We say that *condition* H_{-1} holds if the following two inequalities

$$\left| \int \overline{\psi}(\xi) u(\xi)\, d\overline{\nu_\alpha}(\xi) \right| \le C \sqrt{D(u)} \tag{21}$$

$$\left| \int u(\xi)(\overline{\Omega} u_\lambda)(\xi)\, d\overline{\nu_\alpha}(\xi) \right| \le C \sqrt{D(u)} \tag{22}$$

hold for all cylinder functions u, for all $\lambda > 0$ and for some positive $C > 0$, where u_λ is defined via the equation $\lambda u_\lambda - \overline{\Omega} u_\lambda = \overline{\psi}$.

Liggett proves that condition H_{-1} ensures that the limit in Theorem 17 is a (possibly degenerate) Gaussian random variable: We define $E(a)$ for each measurable function a on $\{0, 1\}^{\mathbb{Z}^d} \cap \{\xi : \xi(0) = 1\}$ as follows:

$$\begin{aligned} E(a) = \frac{1}{2} \Bigg[\frac{1}{4} \sum_{x \in \{\pm 1, \pm 2\}} & \int (x + a(\tau_x \xi) - a(\xi))^2 (1 - \xi(x))\, d\overline{\nu_\alpha}(\xi) \\ & + 2 D_{ex}(a) \Bigg]. \end{aligned}$$

Our variant of condition (20) is the following condition:

$$\left| \int \psi(\xi) u(\xi)\, d\overline{\nu_\alpha}(\xi) \right| \leq C\sqrt{E(u)} \tag{23}$$

holds for all cylinder functions u, where C is some positive multiple of $\sqrt{\alpha(1-\alpha)}$ which does not depend on α. We also prove that condition H_{-1} holds in our model.

5. Main result

For a fixed $0 < \alpha < 1$ let ν_α be the product measure on $\mathbb{Z}$ with constant marginals α, let $\overline{\nu_\alpha} = \nu_\alpha\{\cdot|\xi(0) = 1\}$ and let ξ denote an element of $\{0,1\}^{\mathbb{Z}}$. Let $\mathcal{C}$ denote the real-valued functions on $\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}$ which depend on a finite set of coordinates and let $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ denote the space of continuous real-valued functions on $\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}$. For each $\xi \in \{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}$ we define

$$\xi_{x,y}(a) = \begin{cases} \xi(y) & \text{if } a = x \\ \xi(x) & \text{if } a = y \\ \xi(a) & \text{otherwise} \end{cases} \tag{24}$$

and

$$(\tau_x \xi)(a) = \begin{cases} 0 & \text{if } a = -x \\ 1 & \text{if } a = 0 \\ \xi(a+x) & \text{otherwise} \end{cases} \tag{25}$$

We define for each $a \in \mathcal{C}$, $La = L_{sh}a + L_{ex}a$ (sh and ex stand for the shift and exchange portions of L respectively), where

$$\begin{aligned} \big(L_{sh}a\big)(\xi) &= \frac{1}{4} \sum_{x \in \{\pm 1, \pm 2\}} (a(\tau_x \xi) - a(\xi))(1 - \xi(x)) \\ \big(L_{ex}a\big)(\xi) &= \frac{1}{2} \sum_{x \neq 0, -1} (a(\xi_{x,x+1}) - a(\xi))\xi(x)(1 - \xi(x+1)) \\ &\quad + \frac{1}{2} \sum_{x \neq 0, 1} (a(\xi_{x,x-1}) - a(\xi))\xi(x)(1 - \xi(x-1)). \end{aligned}$$

By the Stone-Weierstrass theorem and Lemmas 1 and 2, the closures of L and L_{ex} in $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, which by abuse of notation we also denote by L and L_{ex} respectively, are Markov pregenerators. We note that we can define $L_{sh}a$ for all $a \in C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ since it is a bounded operator. Similarly, the extension of L_{sh} to $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, which we also denote by L_{sh}, is also a Markov pregenerator and in fact is a Markov generator by Lemma 3 since it is bounded.

In order to prove that L is a Markov generator we need the following definition and two results from Gustafson and Liggett ([12], Theorem 2 and [20], Theorem 2.2). The connection of Markov generators to Theorem 18 was pointed out by Ferrari ([9]).

Definition 15. We call a bounded operator A on a Banach space X dissipative if it satisfies $||f - \lambda A f|| \geq ||f||$ for all $\lambda > 0$ and $f \in X$.

Theorem 18. (Gustafson's perturbation Theorem) Let A be the infinitesimal generator of a contraction semigroup on the Banach space X and let B be a bounded dissipative operator on X. Then $A + B$ is the infinitisemal generator of a contraction semigroup on X.

Theorem 19. Let A be a (possibly unbounded) operator on a Banach space X that takes the form $A = \sum_{n=1}^{\infty} M_n U_n$, where M_n and U_n are a sequence of bounded operators and for each n the finite sum $A = \sum_{i=1}^{n} M_i U_i$ is dissipative. Let μ_n be positive numbers which satisfy $||M_n|| \le \mu_n$ and let $\mathcal{C}_1 = \{f \in X : \sum_{n=1}^{\infty} ||U_n f|| \mu_n < \infty\}$. If $\mathcal{C}_1$ is dense in X and there exists a positive N such that for all n

$$\sum_{k=1}^{\infty} \mu_k \gamma(U_k, U_n) \le N \tag{26}$$

and

$$\sum_{k=1}^{\infty} \mu_k ||[U_k, M_n]|| \le N \mu_n, \tag{27}$$

where $[B, C] = BC - CB$ and $\gamma(B, C) = \sup_f \frac{||[B,C]f||}{||Bf|| + ||Cf||}$, then $\mathcal{R}(I - \lambda A) = X$ for $\lambda > 0$ sufficiently small (recall that $\mathcal{R}$ is used to denote the range of the operator).

Lemma 5. L_{sh} is bounded on $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ and dissipative.

Proof. The proof follows the proof of Lemma 3.1 in [20]. The boundedness is clear. To prove dissipativity, for $f \in C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, by compactness there exists an ξ such that either $f(\xi) = \max\{f(\xi) : \xi \in \{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\} = ||f||_\infty$ or $(-f)(\xi) = \max\{(-f)(\xi) : \xi \in \{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\} = ||(-f)||_\infty$ holds. We'll prove the result for the first case (the second case is similar and thus omitted). Let $\lambda > 0$.

$$\begin{aligned} ||f - \lambda L_{sh} f||_\infty \ge & f(\xi) - \lambda (L_{sh} f)(\xi) \\ & = f(\xi) + \frac{\lambda}{4} \sum_{x \in \{\pm 1, \pm 2\}} (f(\xi) - f(\tau_x \xi))(1 - \xi(x)) \\ \ge & f(\xi) = ||f||_\infty. \end{aligned}$$

□

Lemma 6. $A = L_{ex}$ satisfies the conditions of Theorem 19 with respect to the Banach space $X = C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$.

Proof. The proof follows the proofs of Lemmas 3.1 and 3.6 and Theorem 3.7 in [20]. First note that, for all bounded operators B and C on X, the following inequalities hold: $||[B, C]|| \le 2||B||||C||$ and $\gamma(B, C) \le ||B|| + ||C||$. We define

$$\begin{aligned} (U_{a,b} f)(\xi) &= f(\xi_{a,b}) - f(\xi) \\ (M_{a,b} f)(\xi) &= \frac{1}{2} f(\xi) \xi(a)(1 - \xi(b)). \end{aligned}$$

Thus, $L_{ex} = \sum_{a \ne 0,-1} M_{a,a+1} U_{a,a+1} + \sum_{a \ne 0,1} M_{a,a-1} U_{a,a-1}$. The proof of the dissipativity of the finite sums is the same as the proof of Lemma 5. $||M_{a,b}|| \le \frac{1}{2}$ so we can set $\mu_{a,b} = \frac{1}{2}$ and also $||U_{a,b}|| \le 2$. In order to prove the density of $\mathcal{C}_1$, we note that $\mathcal{C} \subseteq \mathcal{C}_1$ and $\mathcal{C}$ is dense in $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ by the Stone-Weierstrass theorem. If $\{a, b\} \cap \{c, d\} = \emptyset$, then $U_{a,b}$ and $U_{c,d}$ commute and also $U_{a,b}$ and $M_{c,d}$ commute, so $\gamma(U_{a,b}, U_{c,d}) = 0$ and $||[U_{a,b}, M_{c,d}]|| = 0$. We prove the boundedness of the sum in (26) for $U_{a,a+1}$, $a \ne 0, -1$ (the case $U_{a,a-1}$, $a \ne 0, 1$ is similar). Since $U_{a,a+1}$ trivially commutes with itself and since $\gamma(U_{a,b}, U_{c,d}) = 0$ whenever $\{a, b\} \cap \{c, d\} = \emptyset$, the sum only contains at most two terms: $\frac{1}{2}\gamma(U_{a-1,a}, U_{a,a+1})$ (whenever $a \ne 1$) and $\frac{1}{2}\gamma(U_{a+1,a+2}, U_{a,a+1})$ (whenever $a \ne -2$), and thus since $\gamma(U_{a,b}, U_{c,d}) \le ||U_{a,b}|| + ||U_{c,d}|| \le 4$, the sum is bounded by 4. We now prove the boundedness of the sum (27) for $M_{a,a+1}$, $a \neq 0, -1$ (the case $M_{a,a-1}$, $a \neq 0, 1$ is similar). In this case, the sum contains at most three terms: $\frac{1}{2}|||[U_{a-1,a}, M_{a,a+1}]|||$ (whenever $a \neq 1$), $\frac{1}{2}|||[U_{a,a+1}, M_{a,a+1}]|||$ and $\frac{1}{2}|||[U_{a+1,a+2}, M_{a,a+1}]|||$ (whenever $a \neq -2$), and thus, since $|||[U_{a,b}, M_{c,d}]||| \leq 2||U_{a,b}||||M_{c,d}|| \leq 2$, the sum is at most 6, so we can set $N = 4$. $\square$

We conclude from Lemmas 5 and 6 and Theorem 18 that

Lemma 7. L is a Markov generator of a contraction semigroup on $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$.

Let $\mathcal{C}_2$ denote the subset of functions in $C\big((\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}) \cap \{\xi : \xi(0) = 1\}\big)$ which depend on x and a finite number of coordinates of ξ. For each $a \in \mathcal{C}_2$ we define $L_1 a = L_{1,ex} a + L_{1,sh} a$, where $L_{1,ex} a$ and $L_{1,sh} a$ are given by

$$\begin{aligned}
\big(L_{1,ex} a\big)(x,\xi) &= \frac{1}{2} \sum_{y \neq 0,-1} (a(x, \xi_{y,y+1}) - a(x,\xi))\xi(y)(1 - \xi(y+1)) \\
&\quad + \frac{1}{2} \sum_{y \neq 0,1} (a(x, \xi_{y,y-1}) - a(x,\xi))\xi(y)(1 - \xi(y-1)) \\
\big(L_{1,sh} a\big)(x,\xi) &= \frac{1}{4} \sum_{y \in \{\pm 1, \pm 2\}} (a(x+y, \tau_y \xi) - a(x,\xi))(1 - \xi(y)).
\end{aligned}$$

Before stating the version of Lemma 7 for L_1 we recall the locally compact version of the Stone-Weierstrass theorem, which can be found in [6].

Theorem 20. Let X be a locally-compact Hausdorff space and let $C(X)$ denote the real-valued continuous functions on X which vanish at infinity. A subalgebra A of $C(X)$ is said to vanish nowhere if for all $x \in X$ there exists a $a \in A$ such that $a(x) \neq 0$. and is said to separate points if for every two different points $x, y \in X$ there exists a function $a \in A$ such that $a(x) \neq a(y)$. Then A is dense in $C(x)$ with respect to the supremum norm if and only if it separates points and vanishes nowhere.

Once more, we let L_1, $L_{1,ex}$ and $L_{1,sh}$ denote, by abuse of notation, the closures of L_1, $L_{1,ex}$ and $L_{1,sh}$ in $C\big((\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}) \cap \{\xi : \xi(0) = 1\}\big)$ respectively.

Lemma 8. L_1 is a Markov generator on $C\big((\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}) \cap \{\xi : \xi(0) = 1\}\big)$.

Proof. The proof that L_1 is a generator is similar to the proof for L. The proof that $L_{1,sh}$ is dissipative and bounded is the same as Lemma 5 and the proof of Lemma 6 for $L_{1,ex}$ is almost the same with a small difference in proving the density of $\mathcal{C}_1$. In this case $\mathcal{C}_2 \subseteq \mathcal{C}_1$, and $\mathcal{C}_2$ is dense in $C\big((\mathbb{Z} \times \{0,1\}^{\mathbb{Z}}) \cap \{\xi : \xi(0) = 1\}\big)$ by the locally compact version of the Stone-Weierstrass theorem which can be verified by observing the family of functions $a_b(x,\xi) = \frac{1}{x^2+1}\xi(b)$ and $a(x,\xi) = \frac{1}{x^2+1}$. We also note that the verification that $L_{1,ex}$ is a Markov pregenerator is slightly different. To verify condition (i), we take a sequence of functions which depend only on x, $f_n(x,\xi) = f_n(x)$ such that $0 \leq f_n(x) \leq 1$ for all x, $f_n(x) = 1$ whenever $|x| \leq n$ and $f_n(x) \to 0$ as $|x| \to \infty$ and note that clearly $L_{1,ex} f_n = 0$ for all n. To show condition (iii) for $L_{1,ex}$, we note that since for each $f \in \mathcal{C}_2$ and $\lambda \geq 0$, $f - \lambda L_{1,ex} f$ vanishes at infinity, we have $\inf_{\{(x,\xi):\xi(x)=1\}}(f - \lambda L_{1,ex} f)(x,\xi) \leq 0$, so if f only takes non-negative values then, since f vanishes at infinity, $\inf_{\{(x,\xi):\xi(x)=1\}} f(x,\xi) = 0$, so in this case $\inf_{\{(x,\xi):\xi(x)=1\}}(f - \lambda L_{1,ex} f)(x,\xi) \leq \inf_{\{(x,\xi):\xi(x)=1\}} f(x,\xi)$. If f takes a negative value a, then since $A = \{(x,\xi) : |f(x,\xi)| \geq |a|\}$ is compact, f attains a minimum on A at some (x^0, ξ^0) which is a global minimum. Thus,

$$\begin{aligned} f(x^0,\xi^0) - \lambda(L_{1,ex}f)(x^0,\xi^0) &= f(x^0,\xi^0) + \lambda\frac{1}{2}\sum_{y\neq 0,-1}(f(x^0,\xi^0) - f(x^0,\xi^0_{y,y+1}))\xi(y)(1-\xi(y+1)) \\ &\quad + \lambda\frac{1}{2}\sum_{y\neq 0,1}(f(x^0,\xi^0) - f(x^0,\xi^0_{y,y-1}))\xi(y)(1-\xi(y-1)) \\ &\leq f(x^0,\xi^0) = \inf_{\{(x,\xi):\xi(x)=1\}} f(x,\xi) \end{aligned}$$

so in this case we also have $\inf_{\{(x,\xi):\xi(x)=1\}}(f - \lambda L_{1,ex}f)(x,\xi) \leq \inf_{\{(x,\xi):\xi(x)=1\}} f(x,\xi)$ which completes the proof of condition (iii). □

We note that the verification that $L_{1,sh}$ is a Markov pregenerator is similar to $L_{1,ex}$ and thus, since it is bounded, by Lemma 3 it is also a Markov generator.

Let ξ_t be the process governed by L with initial configuration distributed according to $\overline{\nu_\alpha}$ and let X_t denote the position of the tagged particle. Our main result is the following:

Theorem 1. The position of the tagged particle, X_t, satisfies $X_t/\sqrt{t}$ converges in distribution to a normal random variable with non-zero variance and zero mean.

Our proof is similar to Liggett's proof of Theorem 4.55 in section 4 in part III of Liggett ([21]). To follow this strategy we need to show that L and L_1 are generators (which we've already done), that ξ_t is stationary and ergodic, that condition H_{-1} (inequalities (21) and (22)) is satisfied and to prove that condition 23, our variant of condition (20), is satisfied (condition (20) is not satisfied in our model which can be verified by choosing the function $u(\xi) = \sum_{i=1}^n (1 - \frac{i-1}{n})\xi(i)$ since $D_{ex}(u)$ is of order $\frac{1}{n}$ while $\int \psi u \, d\overline{\nu_\alpha}$ is of constant order).

Definition 16. A probability measure $\mu \in \mathcal{P}$ is said to be *reversible* for the process with semigroup $S(t)$ if

$$\int fS(t)g \, d\mu = \int gS(t)f \, d\mu$$

holds for all $f, g \in C(X)$ and all $t \geq 0$.

By plugging in $g = 1$ or a sequence $g_n \nearrow 1$ and writing $f = \max\{f, 0\} - \max\{-f, 0\}$ and applying monotone convergence, we conclude that a reversible measure is also stationary.

Lemma 9. ξ_t is stationary and L satisfies $\int \big(Lf\big)g \, d\overline{\nu_\alpha} = \int f\big(Lg\big) \, d\overline{\nu_\alpha}$ for all $f, g \in \mathcal{C}$.

Proof. We note that ξ_t is stationary if $\overline{\nu_\alpha}$ is stationary, since a Markov process starting from a stationary distribution is stationary. By Proposition 5.3 in chapter 2 in Liggett ([23]) the measure $\overline{\nu_\alpha}$ is reversible iff $\int \big(Lf\big)(\xi)g(\xi) \, d\overline{\nu_\alpha}(\xi) = \int f(\xi)\big(Lg\big)(\xi) \, d\overline{\nu_\alpha}(\xi)$ holds for all $f, g \in \mathcal{C}$, which we shall prove by using the following two equalities

$$\int f(\xi_{x,y})g(\xi)\xi(x)(1-\xi(y)) \, d\overline{\nu_\alpha}(\xi) = \int f(\xi)g(\xi_{x,y})\xi(y)(1-\xi(x)) \, d\overline{\nu_\alpha}(\xi)$$
$$\int f(\tau_x\xi)g(\xi)(1-\xi(x)) \, d\overline{\nu_\alpha}(\xi) = \int f(\xi)g(\tau_{-x}\xi)(1-\xi(-x)) \, d\overline{\nu_\alpha}(\xi)$$

which follow from the fact that the mapping $\xi \to \xi_{x,y}$ is $\overline{\nu_\alpha}$ measure preserving and the mapping $\xi \to \tau_x\xi$ sends the measure $(1-\xi(x)) \, d\overline{\nu_\alpha}(\xi)$ to $(1-\xi(-x)) \, d\overline{\nu_\alpha}(\xi)$. We show that $\overline{\nu_\alpha}$ is reversible for the processes generated by L_{sh} and L_{ex}, i.e. L_{sh} and L_{ex} satisfy for all $f, g \in \mathcal{C}$ the equalities $\int \big(L_{sh}f\big)(\xi)g(\xi) \, d\overline{\nu_\alpha}(\xi) = \int f(\xi)\big(L_{sh}g\big)(\xi) \, d\overline{\nu_\alpha}(\xi)$ and $\int \big(L_{ex}f\big)(\xi)g(\xi) \, d\overline{\nu_\alpha}(\xi) = \int f(\xi)\big(L_{ex}g\big)(\xi) \, d\overline{\nu_\alpha}(\xi)$ respectively. We start with the proof for L_{sh}

$$
\begin{aligned}
&\int \big(L_{sh} f\big)(\xi) g(\xi)\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\tau_x \xi) g(\xi)(1 - \xi(x)))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\xi) g(\xi)(1 - \xi(x)))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

Applying the mapping $\xi \to \tau_x \xi$ yields:

$$
\begin{aligned}
&\frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\tau_x \xi) g(\xi)(1 - \xi(x)))\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\xi) g(\tau_{-x} \xi)(1 - \xi(-x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

By changing the summation variable $x \to -x$ we obtain:

$$
\begin{aligned}
&\frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\tau_x \xi) g(\xi)(1 - \xi(x)))\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\xi) g(\tau_x \xi)(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
&\int \big(L_{sh} f\big)(\xi) g(\xi)\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\xi) g(\tau_x \xi)(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{4} \int \sum_{x \in \{\pm 1, \pm 2\}} f(\xi) g(\xi)(1 - \xi(x)))\, d\overline{\nu_\alpha}(\xi) \\
&= \int \big(L_{sh} g\big)(\xi) f(\xi)\, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

We now prove $\int \big(L_{ex} f\big)(\xi) g(\xi)\, d\overline{\nu_\alpha}(\xi) = \int f(\xi) \big(L_{ex} g\big)(\xi)\, d\overline{\nu_\alpha}(\xi)$.

$$
\begin{aligned}
&\int \big(L_{ex} f\big)(\xi) g(\xi)\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{2} \int \sum_{x \neq 0, -1} f(\xi_{x,x+1}) g(\xi) \xi(x)(1 - \xi(x+1))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{2} \int \sum_{x \neq 0, -1} f(\xi) g(\xi) \xi(x)(1 - \xi(x+1))\, d\overline{\nu_\alpha}(\xi) \\
&+ \frac{1}{2} \int \sum_{x \neq 0, 1} f(\xi_{x,x-1}) g(\xi) \xi(x)(1 - \xi(x-1))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{2} \int \sum_{x \neq 0, 1} f(\xi) g(\xi) \xi(x)(1 - \xi(x-1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

By applying the $\overline{\nu_\alpha}$ measure preserving change of variable $\xi \to \xi_{x,x+1}$:

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi_{x,x+1})g(\xi)\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi)g(\xi_{x,x+1})\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

Similarly, by applying the change of variable $\xi \to \xi_{x,x-1}$:

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,1} f(\xi_{x,x-1})g(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{2}\int \sum_{x\neq 0,1} f(\xi)g(\xi_{x,x-1})\xi(x-1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

By changing the summation variable $x \to x-1$:

$$\begin{aligned}
&-\frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi)g(\xi)\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi) \\
&= -\frac{1}{2}\int \sum_{x\neq 0,1} f(\xi)g(\xi)\xi(x-1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

Similarly, by changing the summation variable $x \to x+1$:

$$\begin{aligned}
&-\frac{1}{2}\int \sum_{x\neq 0,1} f(\xi)g(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi) \\
&= -\frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi)g(\xi)\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

Thus, plugging in all the expressions we obtained yields:

$$\begin{aligned}
&\int \big(L_{ex} f\big)(\xi)g(\xi)\, d\overline{\nu_\alpha}(\xi) \\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi)g(\xi_{x,x+1})\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{2}\int \sum_{x\neq 0,1} f(\xi)g(\xi)\xi(x-1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi) \\
&+ \frac{1}{2}\int \sum_{x\neq 0,1} f(\xi)g(\xi_{x,x-1})\xi(x-1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi) \\
&- \frac{1}{2}\int \sum_{x\neq 0,-1} f(\xi)g(\xi)\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi) \\
&= \int \big(L_{ex} g\big)(\xi)f(\xi)\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

completing the proof. □

By Lemma 9, the measure $\overline{\nu_\alpha}$ is stationary for the processes with Markov generators L,L_{ex} and L_{sh}. Thus, by Proposition 4.1 in chapter IV in [23], the extension of $S(t)$, the Markov semigroup generated by L, to $\mathcal{L}^2(\overline{\nu_\alpha}) = \{f : \{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\} \to \mathbb{R} : \int f^2\, d\overline{\nu_\alpha} < \infty\}$, which we denote by $\overline{S}(t)$, is a Markov contraction semigroup and the generator associated with $\overline{S}(t)$, defined for each a in a dense subset of $\mathcal{L}^2(\overline{\nu_\alpha})$ as $\lim_{t\downarrow 0} \frac{\overline{S}(t)a-a}{t}$

(here the limit is taken in $\mathcal{L}^2(\overline{\nu_\alpha})$) is the closure in $\mathcal{L}^2(\nu_\alpha)$ of L (note that the generator associated with $\overline{S}(t)$ is trivially an extension of L, since a series that converges with respect to the supremum norm also converges in $\mathcal{L}^2(\nu_\alpha)$ to the same limit). Similarly we extend L_{ex} and L_{sh} to $\mathcal{L}^2(\overline{\nu_\alpha})$. By abuse of notation we now use L, L_{ex}, L_{sh}, $S(t)$, $\mathcal{D}(L)$ and $\mathcal{D}(L_{ex})$ to denote the extended generators, semigroup and domains respectively (we note that $\mathcal{D}(L_{sh}) = \mathcal{L}^2(\overline{\nu_\alpha})$). Thus, $La = \lim_{t\downarrow 0} \frac{S(t)a-a}{t}$ for each a belonging to $\mathcal{D}(L)$, a dense subset of $\mathcal{L}^2(\overline{\nu_\alpha})$. Recall that we originally constructed L as an operator on $C(X)$ as the closure of its values on $\mathcal{C}$, so we obtain that $\mathcal{C}$ is a core of L in the following sense.

Definition 17. Suppose Ω is a Markov generator. A linear subspace D of $\mathcal{D}(\Omega)$ is called a *core* for Ω if Ω is the closure of its restriction to D.

Lemma 10. $\int \big(Lf\big) g \, d\overline{\nu_\alpha} = \int f \big(Lg\big) \, d\overline{\nu_\alpha}$ holds for all $f, g \in \mathcal{D}(L)$.

Proof. Let $f, g \in \mathcal{D}(L)$. By the core property of $\mathcal{C}$ we can find $f_n, g_n \in \mathcal{C}$ such that $f_n \to f$, $Lf_n \to Lf$, $g_n \to g$ and $Lg_n \to Lg$, where the sequences convergence in $\mathcal{L}^2(\overline{\nu_\alpha})$. By applying the Cauchy-Schwartz inequality we obtain:

$$\begin{aligned}
&\left| \int \big(Lf\big) g \, d\overline{\nu_\alpha} - \int \big(Lf_n\big) g_n \, d\overline{\nu_\alpha} \right| \\
&\le \left| \int \big(Lf\big) g \, d\overline{\nu_\alpha} - \int \big(Lf_n\big) g \, d\overline{\nu_\alpha} \right| + \left| \int \big(Lf_n\big) g \, d\overline{\nu_\alpha} - \int \big(Lf_n\big) g_n \, d\overline{\nu_\alpha} \right| \\
&\le \left(\int \big(Lf - Lf_n\big)^2 \, d\overline{\nu_\alpha} \right)^{1/2} \left(\int g^2 \, d\overline{\nu_\alpha} \right)^{1/2} + \left(\int \big(g - g_n\big)^2 \, d\overline{\nu_\alpha} \right)^{1/2} \left(\int \big(Lf_n\big)^2 \, d\overline{\nu_\alpha} \right)^{1/2} \to 0
\end{aligned}$$

as $n \to \infty$ and thus $\int \big(Lf\big) g \, d\overline{\nu_\alpha} = \lim_{n\to\infty} \int \big(Lf_n\big) g_n \, d\overline{\nu_\alpha}$ and similarly $\int \big(Lg\big) f \, d\overline{\nu_\alpha} = \lim_{n\to\infty} \int \big(Lg_n\big) f_n \, d\overline{\nu_\alpha}$. By Lemma 9, for all n, $\int \big(Lf_n\big) g_n \, d\overline{\nu_\alpha} = \int f_n \big(Lg_n\big) \, d\overline{\nu_\alpha}$ holds and thus

$$\begin{aligned}
\int \big(Lf\big) g \, d\overline{\nu_\alpha} &= \lim_{n\to\infty} \int \big(Lf_n\big) g_n \, d\overline{\nu_\alpha} \\
&= \lim_{n\to\infty} \int f_n \big(Lg_n\big) \, d\overline{\nu_\alpha} \\
&= \int f \big(Lg\big) \, d\overline{\nu_\alpha}
\end{aligned}$$ □

Similarly we obtain

Lemma 11. $\int \big(L_{ex} f\big) g \, d\overline{\nu_\alpha} = \int f \big(L_{ex} g\big) \, d\overline{\nu_\alpha}$ holds for all $f, g \in \mathcal{D}(L_{ex})$.

and

Lemma 12. $\int \big(L_{sh} f\big) g \, d\overline{\nu_\alpha} = \int f \big(L_{sh} g\big) \, d\overline{\nu_\alpha}$ holds for all $f, g \in \mathcal{L}^2(\overline{\nu_\alpha})$.

Let $D(a)$ be the Dirichlet form of a measurable function a given by $D(a) = D_{sh}(a) + D_{ex}(a)$, where

$$\begin{aligned}
D_{sh}(a) &= \frac{1}{8} \sum_{x \in \{\pm 1, \pm 2\}} \int (a(\tau_x \xi) - a(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi) \\
D_{ex}(a) &= \frac{1}{4} \sum_{x \neq 0, -1} \int (a(\xi_{x,x+1}) - a(\xi))^2 \xi(x)(1 - \xi(x+1)) \, d\overline{\nu_\alpha}(\xi) \\
&\quad + \frac{1}{4} \sum_{x \neq 0, 1} \int (a(\xi_{x,x-1}) - a(\xi))^2 \xi(x)(1 - \xi(x-1)) \, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

We also define $E(a)$ as follows:

$$E(a) = \frac{1}{2}\Big[\frac{1}{4}\sum_{x\in\{\pm 1,\pm 2\}}\int (x + a(\tau_x\xi) - a(\xi))^2(1-\xi(x))\, d\overline{\nu_\alpha}(\xi) + 2D_{ex}(a)\Big]$$

We note that the following two lemmas are somewhat analogous to integration by parts, since if we replace the generators by the second derivative and use a to denote continuous real-valued functions, then the right hand side is of the form $-\int a''(s)a(s)\, ds$ and the left hand side is of the form $\int (a'(s))^2\, ds$.

Lemma 13. $D_{sh}(a) = \int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{C}$.

Proof. By plugging in $L_{sh}a$ into the integral we obtain:

$$\int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi) = \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))a(\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)$$

By writing $a(\xi) = a(\xi) - a(\tau_x\xi) + a(\tau_x\xi)$ we obtain

$$\begin{aligned}\int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi) &= \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))^2(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\\ &+ \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))a(\tau_x\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

By applying the maping $\xi \to \tau_x\xi$:

$$\begin{aligned}&\frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))a(\tau_x\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\tau_{-x}\xi) - a(\xi))a(\xi)(1-\xi(-x)))\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

By the change of summation variable $x \to -x$:

$$\begin{aligned}&\frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))a(\tau_x\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\tau_x\xi) - a(\xi))a(\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\\ &= -\frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))a(\xi)(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

Thus,

$$\begin{aligned}\int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi) &= \frac{1}{4}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\xi) - a(\tau_x\xi))^2(1-\xi(x)))\, d\overline{\nu_\alpha}(\xi)\\ &- \int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

Rearranging the terms yields:

$$\begin{aligned}
&\int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{8}\int \sum_{x\in\{\pm 1,\pm 2\}} (a(\tau_x\xi) - a(\xi))^2(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)\\
&= D_{sh}(a)
\end{aligned}$$

□

Lemma 14. $D_{ex}(a) = \int \big(-L_{ex}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{C}$.

Proof. Plugging in $L_{ex}a$ into the integral yields:

$$\begin{aligned}
&\int \big(-L_{ex}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi)\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&+ \frac{1}{2}\int \sum_{x\neq 0,1} (a(\xi) - a(\xi_{x,x-1}))a(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

We write $a(\xi) = a(\xi) - a(\xi_{x,x+1}) + a(\xi_{x,x+1})$ to obtain:

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi)\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))^2\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&+ \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi^{x,x+1})\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

By the change of variable $\xi \to \xi_{x,x+1}$:

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi_{x,x+1})\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi_{x,x+1}) - a(\xi))a(\xi)\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

By changing the summation variable $x \to x-1$ and noting that $\xi_{x-1,x} = \xi_{x,x-1}$ we obtain:

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi_{x,x+1})\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{2}\int \sum_{x\neq 0,1} (a(\xi_{x,x-1}) - a(\xi))a(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)\\
&= -\frac{1}{2}\int \sum_{x\neq 0,1} (a(\xi) - a(\xi_{x,x-1}))a(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

And thus

$$\begin{aligned}
&\frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))a(\xi)\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi) - a(\xi_{x,x+1}))^2\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\
&- \frac{1}{2}\int \sum_{x\neq 0,1} (a(\xi) - a(\xi_{x,x-1}))a(\xi)\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

Rearranging the terms yields:

$$\begin{aligned}&\int \big(-L_{ex}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi)-a(\xi_{x,x+1}))^2\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

Applying the change of variable $\xi \to \xi_{x,x+1}$ yields:

$$\begin{aligned}&\frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi)-a(\xi_{x,x+1}))^2\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{2}\int \sum_{x\neq 0,-1} (a(\xi_{x,x+1})-a(\xi))^2\xi(x+1)(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{2}\int \sum_{x\neq 0,1} (a(\xi)-a(\xi_{x,x-1}))^2\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)\end{aligned}$$

Thus:

$$\begin{aligned}&\int \big(-L_{ex}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)\\ &= \frac{1}{4}\int \sum_{x\neq 0,1} (a(\xi)-a(\xi_{x,x-1}))^2\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)\\ &+ \frac{1}{4}\int \sum_{x\neq 0,-1} (a(\xi)-a(\xi_{x,x+1}))^2\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\ &= D_{ex}(a)\end{aligned}$$

□

We are now ready to prove the following result.

Lemma 15. $D(a) = \int \big(-La\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{D}(L)$.

Proof. By Lemmas 13 and 14, we obtain that $D(a) = \int \big(-La\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{C}$. Let $D(a,b)$ be defined as

$$\begin{aligned}D(a,b) = \frac{1}{2}\Bigg[&\frac{1}{4}\sum_{x\in\{\pm1,\pm2\}} \int (a(\tau_x\xi)-a(\xi))(b(\tau_x\xi)-b(\xi))(1-\xi(x))\, d\overline{\nu_\alpha}(\xi)\\ &+ \frac{1}{2}\sum_{x\neq 0,-1}\int (a(\xi_{x,x+1})-a(\xi))(b(\xi_{x,x+1})-b(\xi))\xi(x)(1-\xi(x+1))\, d\overline{\nu_\alpha}(\xi)\\ &+ \frac{1}{2}\sum_{x\neq 0,1}\int (a(\xi_{x,x-1})-a(\xi))(b(\xi_{x,x-1})-b(\xi))\xi(x)(1-\xi(x-1))\, d\overline{\nu_\alpha}(\xi)\Bigg]\end{aligned}$$

so that $D(a,a) = D(a)$. We define $R(a) = \int \big(-La\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$. The proof follows the proof in chapter IV of Liggett ([23], proposition 4.1 (page 205)). Recall that a convergent sequence in $\mathcal{L}^2(\overline{\nu_\alpha})$ has a pointwise almost everywhere convergent subsequence, and thus, by the core property of $\mathcal{C}$, we conclude that for each $a \in \mathcal{D}(L)$ we can find a sequence a_n of elements in $\mathcal{C}$ such that $a_n \to a$ pointwise almost everywhere and in $\mathcal{L}^2(\overline{\nu_\alpha})$, and $La_n \to La$ in $\mathcal{L}^2(\overline{\nu_\alpha})$. Thus, by Fatou's lemma, $D(a) \leq \liminf_{n\to\infty} D(a_n)$, and by Lemma 10, $R(a) = \lim_{n\to\infty} R(a_n)$ and (by Cauchy-Schwartz) $R(a) < \infty$. Since $R(a_n) = D(a_n)$ we conclude that $D(a) \leq R(a)$, and in particular, $D(a-a_n) \leq R(a-a_n)$ and $D(a) < \infty$. By the Cauchy-Schwartz inequality,

$$(R(a-a_n))^2 \leq \int \big(-L(a-a_n)\big)^2(\xi)\, d\overline{\nu_\alpha}(\xi) \int (a-a_n)^2(\xi)\, d\overline{\nu_\alpha}(\xi) \to 0$$

as $n \to \infty$. Thus, $D(a - a_n) \to 0$ as $n \to \infty$. Therefore, by plugging in $a = a - a_n + a_n$ into $D(a)$ and opening the squared terms, we obtain:

$$D(a) = D(a - a_n) + D(a_n) + 2D(a - a_n, a_n).$$

Thus, applying once again Cauchy-Schwartz:

$$D(a - a_n, a_n) \leq \sqrt{D(a - a_n)}\sqrt{D(a_n)} \to 0$$

as $n \to \infty$ and thus $D(a) = \lim_{n\to\infty} D(a_n) = \lim_{n\to\infty} R(a_n) = R(a)$. $\square$

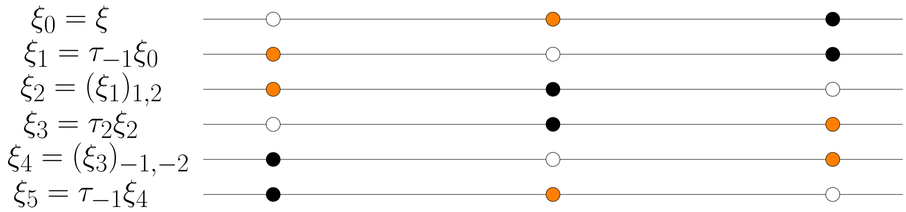
Similarly we obtain the following results.

Lemma 16. $D_{ex}(a) = \int \big(-L_{ex}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{D}(L_{ex})$.

Lemma 17. $D_{sh}(a) = \int \big(-L_{sh}a\big)(\xi)a(\xi)\, d\overline{\nu_\alpha}(\xi)$ for all $a \in \mathcal{L}^2(\overline{\nu_\alpha})$.

Lemma 18. For all $f \in \mathcal{C}$ we have $\int (f(\xi) - f(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) \leq C \min\{E(f), D(f)\}$ and thus for all $f \in \mathcal{D}(L)$ we have $\int (f(\xi) - f(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) \leq CD(f)$.

Proof. Note that if we perform the following 5 steps whenever $\xi(-1) = 0$ and $\xi(1) = 1$ we can move from ξ to $\xi_{-1,1}$ (in the image the orange circle denotes the tagged particle, the black circle a different particle and a white circle a position which is particle free).



If we write

$$\begin{aligned} f(\xi) - f(\xi_{-1,1}) = f(\xi) - f(\xi_1) + 1 \\ + f(\xi_1) - f(\xi_2) \\ + f(\xi_2) - f(\xi_3) - 2 \\ + f(\xi_3) - f(\xi_4) \\ + f(\xi_4) - f(\xi_{-1,1}) + 1 \end{aligned}$$

and apply the inequality $(\sum_{i=1}^5 a_i)^2 \leq 5\sum_{i=1}^5 a_i^2$ we obtain:

$$\begin{aligned} &\int (f(\xi) - f(\xi_{-1,1}))^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \\ &\leq 5\int (f(\xi) - f(\tau_{-1}\xi) + 1)^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \\ &+ 5\int (f(\xi_1) - f((\xi_1)_{1,2}))^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \\ &+ 5\int (f(\xi_2) - f(\tau_2\xi_2) - 2)^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \\ &+ 5\int (f(\xi_3) - f((\xi_3)_{-1,-2}))^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \\ &+ 5\int (f(\xi_4) - f(\tau_{-1}\xi_4) + 1)^2 \xi(1)(1 - \xi(-1))\, d\overline{\nu_\alpha}(\xi) \end{aligned}$$

Applying to the $(i+1)^{st}$ term the $\overline{\nu_\alpha}$ measure-preserving change of variables $\xi_i \to \xi$ yields:

$$\begin{aligned}
&\int (f(\xi) - f(\xi_{-1,1}))^2 \xi(1)(1-\xi(-1))\, d\overline{\nu_\alpha}(\xi) \\
&\le 5 \int (f(\xi) - f(\tau_{-1}\xi) + 1)^2 \xi(1)(1-\xi(-1))\, d\overline{\nu_\alpha}(\xi) \\
&+ 5 \int (f(\xi) - f(\xi_{1,2}))^2 \xi(2)(1-\xi(1))\, d\overline{\nu_\alpha}(\xi) \\
&+ 5 \int (f(\xi) - f(\tau_2\xi) - 2)^2 \xi(1)(1-\xi(2))\, d\overline{\nu_\alpha}(\xi) \\
&+ 5 \int (f(\xi) - f(\xi_{-1,-2}))^2 \xi(-1)(1-\xi(-2))\, d\overline{\nu_\alpha}(\xi) \\
&+ 5 \int (f(\xi) - f(\tau_{-1}\xi) + 1)^2 \xi(-2)(1-\xi(-1))\, d\overline{\nu_\alpha}(\xi)
\end{aligned}$$

and each of the 5 terms is bounded by $5 \times 8 \times E(f)$. Noting that, by applying the measure preserving change of variable $\xi \to \xi_{-1,1}$,

$$\int (f(\xi) - f(\xi_{-1,1}))^2 \xi(1)(1-\xi(-1))\, d\overline{\nu_\alpha}(\xi) = \int (f(\xi) - f(\xi_{-1,1}))^2 \xi(-1)(1-\xi(1))\, d\overline{\nu_\alpha}(\xi)$$

and thus

$$\int (f(\xi) - f(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) = 2\int (f(\xi) - f(\xi_{-1,1}))^2 \xi(1)(1-\xi(-1))\, d\overline{\nu_\alpha}(\xi) \le C \times E(f)$$

The bound with $D(f)$ in place of $E(f)$ follows from an even simpler decomposition of $f(\xi) - f(\xi^{-1,1})$ as a sum of 5 terms (the same decomposition as before without the constants). In order to show that the last inequality holds throughout the domain of L recall that a convergent sequence in $\mathcal{L}^2(\overline{\nu_\alpha})$ has a pointwise almost everywhere convergent subsequence, and thus, by the core property of $\mathcal{C}$, we conclude that for each $f \in \mathcal{D}(L)$ we can find a sequence f_n of elements in $\mathcal{C}$ such that $f_n \to f$ pointwise almost everywhere and in $\mathcal{L}^2(\overline{\nu_\alpha})$, and $Lf_n \to Lf$ in $\mathcal{L}^2(\overline{\nu_\alpha})$. By Fatou's lemma and Lemma 15

$$\begin{aligned}
&\int (f(\xi) - f(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) \\
&= \int \liminf_{n\to\infty} (f_n(\xi) - f_n(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) \\
&\le \liminf_{n\to\infty} \int (f_n(\xi) - f_n(\xi_{-1,1}))^2\, d\overline{\nu_\alpha}(\xi) \\
&\le \liminf_{n\to\infty} CD(f_n) = \liminf_{n\to\infty} C\int (-Lf_n) f_n\, d\overline{\nu_\alpha} \\
&= C\int (-Lf) f\, d\overline{\nu_\alpha} = CD(f)
\end{aligned}$$

which completes the proof. □

Definition 18. We call a permutation π on a countable set S a *finite permutation* if $|s \in S : \pi(s) \neq s| < \infty$ (here $|\cdot|$ denotes the cardinality of the set).

We recall the Hewitt-Savage $0-1$ law which can be found in Durrett's book ([8]).

Theorem 21. Let $\{X_n\}_{n=1}^\infty$ be an i.i.d sequence of real-valued random variables. We define the *exchangeable sigma algebra* $\mathcal{E}$ as the set of events depending on the sequence $\{X_n\}_{n=1}^\infty$ which are invariant under finite permutations of the indices of the sequence $\{X_n\}_{n=1}^\infty$. Then $A \in \mathcal{E} \Rightarrow \mathbb{P}(A) \in \{0,1\}$.

For our next result we need the following result that appears in Sethuraman ([30], Proposition 2.1).

Lemma 19. Let μ be an invariant measure on a compact set X and let $\mathcal{B}$ denote the Borel σ-field on X. Let $\mathbb{P}^\mu$ denote the probability on the path space with initial distribution μ and let $\widehat{f}$ denote the $\mathcal{L}^2(\mu)$ limit of $\frac{1}{t}\int_0^t S(s)f\,ds$. All the statements below are equivalent.

(i) For all $A \in \mathcal{B}$, $S(t)\mathbb{1}_A = \mathbb{1}_A$ μ-a.s. $\Rightarrow \mu(A) \in \{0,1\}$.

(ii) $\mathbb{P}^\mu$ is ergodic: For each $f \in \mathcal{L}^2(\mu)$, $\widehat{f} = \mathbb{E}_\mu[f]$ μ-a.s.

(iii) $\mu \in \mathcal{I}_e$.

Lemma 20. The stationary process ξ_t is ergodic.

Proof. The proof is similar to Theorem 5.3 in chapter 5 in part II of Komorowski's book ([16]). By Theorem 5, the stationary process ξ_t is ergodic iff the initial stationary measure $\overline{\nu_\alpha}$ is extremal. By Lemma 19, $\overline{\nu_\alpha}$ is extremal iff all Borel sets A which satisfy $S(t)\mathbb{1}_A = \mathbb{1}_A$ for all $t \geq 0$ also satisfy $\overline{\nu_\alpha}(A) \in \{0,1\}$.

Let A be a Borel set satisfying $S(t)\mathbb{1}_A = \mathbb{1}_A$ for all $t \geq 0$ and we set $f = \mathbb{1}_A$. Since $f \in \mathcal{L}^2(\overline{\nu_\alpha})$ and satisfies $S(t)f = f$ for all $t \geq 0$ we obtain that $f \in \mathcal{D}(L)$, since $Lf = \lim_{t\downarrow 0}\frac{S(t)f-f}{t} = 0$ (see the paragraph after Lemma 9). Thus by Lemma 15, $D(f) = -\int fLf\,d\overline{\nu_\alpha} = 0$ and thus $\int (f(\xi) - f(\xi_{a,b}))^2\,d\overline{\nu_\alpha}(\xi) = 0$ for $a, b \neq 0$ satisfying $\left|a-b\right| = 1$. By applying Lemma 18, we obtain $\int (f(\xi) - f(\xi_{-1,1}))^2\,d\overline{\nu_\alpha}(\xi) = 0$. Thus, for a.e. ξ (with respect to $\overline{\nu_\alpha}$)

$$f(\xi) = f(\xi_{-1,1}) = f(\xi_{a,b})\,. \tag{28}$$

We note that all transpositions (c,d) with $c, d \in \mathbb{Z}\setminus\{0\}$ can be written as a product of transpositions, where each transposition in the product either takes the form (a,b) with $a, b \neq 0$ and $\left|a-b\right| = 1$ or takes the form $(-1,1)$ (the proof follows from an induction on the distance between c and d). Thus, for every $c, d \in \mathbb{Z}\setminus\{0\}$ we can find a finite sequence ξ_i such that $\xi_0 = \xi$, $\xi_k = \xi_{c,d}$ and for each $0 \leq i \leq k-1$, $\xi_{i+1} = (\xi_i)_{x_i,y_i}$ for x_i, y_i which satisfy one of the following conditions for the transposition (x_i, y_i): $(x_i,y_i) = (-1,1)$ or $(x_i,y_i) = (a,b)$ for some a, b which satisfy $a, b \neq 0$ and $|a-b| = 1$. Thus, by equation (28)

$$f(\xi) - f(\xi_{c,d})) = \sum_{i=0}^{k-1}(f(\xi_i) - f(\xi_{i+1})) = 0 \text{ a.e. } \xi\,.$$

For each finite permutation σ on $\mathbb{Z}$ which fixes 0, we write $f(\sigma\xi) = f((\xi(\sigma(i))_{i\in\mathbb{Z}})$. Since the transpositions (c,d) with $c, d \in \mathbb{Z}\setminus\{0\}$ generate the finite permutations on $\mathbb{Z}$ which fix 0, we conclude by similar reasoning used in proving $f(\xi) = f(\xi_{c,d})$ that for every permutation σ on $\mathbb{Z}$ which fixes 0, $f(\sigma\xi) = f(\xi)$ holds for a.e. ξ. Thus, since there are countably many finite permutations on $\mathbb{Z}$ which fix 0, we conclude that there exists a set B with $\overline{\nu_\alpha}(B) = 1$ such that for each $\xi \in B$, the equality $f(\xi) = f(\sigma\xi)$ holds for all finite permutations σ on $\mathbb{Z}$ which fix 0. We set $C = A \cap B$.

Claim 1. We have $\xi \in C \Rightarrow \sigma\xi \in C$ for all finite permutations σ on $\mathbb{Z}$ which fix 0.

Proof. Let $\xi \in C$ and σ a finite permutation on $\mathbb{Z}$ which fixes 0. Thus $\xi \in A$ and thus $f(\xi) = \mathbb{1}_A(\xi) = 1$. In addition, $\xi \in B$ and thus $\mathbb{1}_A(\sigma\xi) = f(\sigma\xi) = f(\xi) = 1$ so $\sigma\xi \in A$. If we assume $\sigma\xi \notin B$ then there exists a finite permutation τ on $\mathbb{Z}$ which fixes 0 such that $1 = f(\sigma\xi) \neq f(\tau\sigma\xi) = \mathbb{1}_A(\tau\sigma\xi)$. Thus, $\tau\sigma\xi \notin A$. Since a composition of two finite permutations on $\mathbb{Z}$ which fixes 0 is also a finite permutation on $\mathbb{Z}$ which fixes 0, we found a finite permutation $\tau\sigma$ on $\mathbb{Z}$ which fixes 0 such that $\xi \in A$ but $\tau\sigma\xi \notin A$ which contradicts the fact we already proved that for each finite permutation σ on $\mathbb{Z}$ which fixes 0, $\sigma\xi \in A$ so $\sigma\xi \in B$ completing the proof of the claim. $\square$

Let $\nu_{\alpha,1}$ denote the product measure on $\mathbb{Z} \setminus \{0\}$ and Π the projection on $\mathbb{Z} \setminus \{0\}$. We note that from the claim we conclude that $\{(\xi(i))_{i\in\mathbb{Z}\setminus\{0\}} : (\xi(i))_{i\in\mathbb{Z}\setminus\{0\}} \in \Pi(C \cap \{\xi(0) = 1\})\}$ is in the exchangeable sigma field, i.e. $(\xi(i))_{i\in\mathbb{Z}\setminus\{0\}} \in \Pi(C \cap \{\xi(0) = 1\}) \Rightarrow \xi(\sigma(i))_{i\in\mathbb{Z}\setminus\{0\}} \in \Pi(C \cap \{\xi(0) = 1\})$ for every finite permutation on $\mathbb{Z} \setminus \{0\}$. By applying Hewitt Savage 0-1 law to the sequence $(\xi(i))_{i\in\mathbb{Z}\setminus\{0\}}$ we obtain

$$\begin{aligned}\overline{\nu_\alpha}(C) &= \overline{\nu_\alpha}(C \cap \{\xi(0) = 1\})\\ &= \nu_{\alpha,1}(\Pi(C \cap \{\xi(0) = 1\})) = \mathbb{P}(\{\xi(i))_{i\in\mathbb{Z}\setminus\{0\}} \in \Pi(C \cap \{\xi(0) = 1\}\}) \in \{0, 1\}.\end{aligned}$$

Thus $\overline{\nu_\alpha}(C) \in \{0, 1\}$ which completes the proof, since clearly $\overline{\nu_\alpha}(A) = \overline{\nu_\alpha}(C)$. $\square$

By Lemma 3, for all $\lambda > 0$ we have $\mathcal{R}(\lambda I - L) = C\big(\{0, 1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, where here L is the original operator on $C\big(\{0, 1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$. Since the function $\psi(\xi) = \frac{1}{4}\sum_{x\in\{\pm1,\pm2\}} x(1 - \xi(x)) \in C\big(\{0, 1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, we can find a function $h \in C\big(\{0, 1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ which depends on λ such that $\lambda h - Lh = \psi$. This yields the following definition:

Definition 19. We define $u_\lambda \in C\big(\{0, 1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ for $\lambda > 0$ via $\lambda u_\lambda - Lu_\lambda = \psi$ where $\psi(\xi) = \frac{1}{4}\sum_{x\in\{\pm1,\pm2\}} x(1 - \xi(x))$ is called the drift.

Lemmas 21 and 22 together form condition H_{-1} (see Def. 14).

Lemma 21. $\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \le C_1(\alpha)\sqrt{D(u)}$ for all $u \in \mathcal{C}$ with $C_1(\alpha) = \sqrt{\frac{5(1-\alpha)}{4}}$.

Proof. We apply the equality $\int u(\tau_x\xi)(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi) = \int u(\xi)(1 - \xi(-x))\, d\overline{\nu_\alpha}(\xi)$ to obtain:

$$\begin{aligned}&\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right|\\ &= \frac{1}{4}\left|\sum_{x\in\{\pm1,\pm2\}} x\int u(\xi)(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)\right|\\ &= \frac{1}{4}\left|\sum_{x\in\{1,2\}} x\int \big(u(\xi) - u(\tau_x\xi)\big)(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)\right|\end{aligned}$$

Applying the Cauchy-Schwartz inequality yields:

$$\begin{aligned}&\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right|\\ &\le \frac{1}{4}\sqrt{\sum_{x\in\{1,2\}}\int x^2(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)}\sqrt{\sum_{x\in\{1,2\}}\int \big(u(\xi) - u(\tau_x\xi)\big)^2(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)}\\ &= \frac{\sqrt{5(1-\alpha)}}{4}\sqrt{\sum_{x\in\{1,2\}}\int \big(u(\xi) - u(\tau_x\xi)\big)^2(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi)}\end{aligned}$$

Applying

$$\int \big(u(\xi) - u(\tau_x\xi)\big)^2(1 - \xi(x))\, d\overline{\nu_\alpha}(\xi) = \int \big(u(\tau_{-x}\xi) - u(\xi)\big)^2(1 - \xi(-x))\, d\overline{\nu_\alpha}(\xi)$$

yields:

$$\frac{\sqrt{5(1-\alpha)}}{4}\sqrt{\sum_{x\in\{1,2\}}\int\big(u(\xi)-u(\tau_x\xi)\big)^2(1-\xi(x))\,d\overline{\nu_\alpha}(\xi)}$$
$$=\frac{\sqrt{5(1-\alpha)}}{4}\sqrt{\frac{1}{2}\sum_{x\in\{\pm1,\pm2\}}\int\big(u(\xi)-u(\tau_x\xi)\big)^2(1-\xi(x))\,d\overline{\nu_\alpha}(\xi)}$$
$$\le\frac{\sqrt{5(1-\alpha)}}{4}\sqrt{4D(u)}=\sqrt{\frac{5(1-\alpha)}{4}}\sqrt{D(u)}$$

completing the proof. □

Lemma 22. $\left|\int u(\xi)\big(Lu_\lambda\big)(\xi)\,d\overline{\nu_\alpha}(\xi)\right|\le C_2(\alpha)\sqrt{D(u)}$ for all $u\in\mathcal{C}$ with $C_2(\alpha)=\sqrt{\frac{5(1-\alpha)}{4}}=C_1(\alpha)$.

Proof. By Lemma 10, L satisfies $\int a\big(Lb\big)\,d\overline{\nu_\alpha}=\int b\big(La\big)\,d\overline{\nu_\alpha}$ for all $a,b\in\mathcal{D}(L)$ and by Lemma 15, $\int a\big(-La\big)\,d\overline{\nu_\alpha}=D(a)\ge 0$ for all $a\in\mathcal{D}(L)$. Thus we can define a semi-inner product $\langle a,b\rangle=\int a\big(-Lb\big)\,d\overline{\nu_\alpha}$ on $\mathcal{D}(L)$. We recall the equation defining $u_\lambda\in\mathcal{D}(L)$ for $\lambda>0$:

$$\lambda u_\lambda - Lu_\lambda=\psi \tag{29}$$

By Lemma 15, $D(u_\lambda)=\int\big(-Lu_\lambda\big)u_\lambda\,d\overline{\nu_\alpha}$ and thus, since L is an operator on $C\big(\{0,1\}^{\mathbb{Z}}\cap\{\xi:\xi(0)=1\}\big)$, we have $u_\lambda, Lu_\lambda\in C\big(\{0,1\}^{\mathbb{Z}}\cap\{\xi:\xi(0)=1\}\big)$, and thus

$$D(u_\lambda)\le\int||Lu_\lambda|||_\infty||u_\lambda||_\infty\,d\overline{\nu_\alpha}<\infty. \tag{30}$$

Multiplying equation (29) by u_λ and integrating by $d\overline{\nu_\alpha}$ yields:

$$\lambda\int u_\lambda^2\,d\overline{\nu_\alpha}+D(u_\lambda)=\int\psi u_\lambda\,d\overline{\nu_\alpha}\,,$$

and thus by dropping the first term which is non-negative,

$$D(u_\lambda)\le\int\psi u_\lambda\,d\overline{\nu_\alpha}\,. \tag{31}$$

Applying Lemma 21 to u_λ (more precisely to a sequence $f_n\in\mathcal{C}$ such that $f_n\to u_\lambda$ and $Lf_n\to Lu_\lambda$ where the convergence is with respect to $\mathcal{L}^2(\overline{\nu_\alpha})$ and noting that by Lemmas 10 and 15 $D(f_n)=\int f_n(-Lf_n)\,d\overline{\nu_\alpha}\to\int u_\lambda(-Lu_\lambda)\,d\overline{\nu_\alpha}=D(u_\lambda)$) yields

$$\int\psi u_\lambda\,d\overline{\nu_\alpha}\le\sqrt{\frac{5(1-\alpha)}{4}}\sqrt{D(u_\lambda)} \tag{32}$$

Equations (31) and (32) yield

$$D(u_\lambda)\le\sqrt{\frac{5(1-\alpha)}{4}}\sqrt{D(u_\lambda)}$$

and thus, since $D(u_\lambda)<\infty$ by equation (30), we can divide both sides by $\sqrt{D(u_\lambda)}$ to obtain:

$$\sqrt{D(u_\lambda)}\le\sqrt{\frac{5(1-\alpha)}{4}}$$

Applying the Cauchy-Schwartz inequality for $\langle\cdot,\cdot\rangle$ yields for all $u \in \mathcal{C}$:

$$\begin{aligned}\left|\int u(\xi)\big(Lu_\lambda\big)(\xi)\, d\overline{\nu_\alpha}(\xi)\right| &= |\langle u, u_\lambda\rangle| \\ &\leq \sqrt{\langle u, u\rangle}\sqrt{\langle u_\lambda, u_\lambda\rangle} \\ &= \sqrt{D(u)}\sqrt{D(u_\lambda)} \\ &\leq \sqrt{\frac{5(1-\alpha)}{4}}\sqrt{D(u)}\end{aligned}$$

□

Lemma 23. $\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \leq C_3(\alpha)\sqrt{E(u)}$ for all $u \in \mathcal{C}$ with $C_3(\alpha) = C\sqrt{\alpha(1-\alpha)}$.

Proof. Since $\sum_{x\in\{\pm1,\pm2\}} x \int u(\xi)\, d\overline{\nu_\alpha}(\xi) = 0$ we obtain:

$$\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| = \frac{1}{4}\left|\sum_{x\in\{1,2\}} x\int u(\xi)(\xi(-x)-\xi(x))\, d\overline{\nu_\alpha}\right|$$

We note that $\int u(\xi)(\xi(-x)-\xi(x))\, d\overline{\nu_\alpha}(\xi) = \int u(\xi_{-x,x})(\xi(x)-\xi(-x))\, d\overline{\nu_\alpha}(\xi)$ and thus $\int u(\xi)(\xi(-x)-\xi(x))\, d\overline{\nu_\alpha}(\xi) = \frac{1}{2}\int\big(u(\xi)-u(\xi_{-x,x})\big)(\xi(-x)-\xi(x))\, d\overline{\nu_\alpha}(\xi)$ and thus:

$$\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| = \frac{1}{8}\left|\sum_{x\in\{1,2\}} x\int\big(u(\xi)-u(\xi_{-x,x})\big)(\xi(-x)-\xi(x))\, d\overline{\nu_\alpha}(\xi)\right|$$

And thus by inserting the absolute value into the integral:

$$\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \leq \frac{1}{8}\sum_{x\in\{1,2\}} x\int\Big|u(\xi_{-x,x})-u(\xi)\Big|\Big|\xi(-x)-\xi(x)\Big|\, d\overline{\nu_\alpha}(\xi)$$

We note that $\big|\xi(-x)-\xi(x)\big| = \big|\xi(-x)-\xi(x)\big|^2$ since $\big|\xi(-x)-\xi(x)\big| \in \{0,1\}$ for all ξ. Thus, by the Cauchy-Schwartz inequality:

$$\begin{aligned}&\left|\int \psi(\xi)u(\xi)\, d\overline{\nu_\alpha}(\xi)\right| \\ &\leq \frac{1}{8}\sum_{x\in\{1,2\}} x\sqrt{\int\big(u(\xi_{-x,x})-u(\xi)\big)^2\, d\overline{\nu_\alpha}(\xi)}\sqrt{\int\big|\xi(-x)-\xi(x)\big|\, d\overline{\nu_\alpha}(\xi)} \\ &= \frac{\sqrt{2\alpha(1-\alpha)}}{8}\sum_{x\in\{1,2\}} x\sqrt{\int\big(u(\xi_{-x,x})-u(\xi)\big)^2\, d\overline{\nu_\alpha}(\xi)}\end{aligned}$$

By Lemma 18, $\int\big(u(\xi_{-1,1})-u(\xi)\big)^2\, d\overline{\nu_\alpha}(\xi) \leq C\times E(u)$, so the result will follow if we show $\int\big(u(\xi_{-2,2})-u(\xi)\big)^2\, d\overline{\nu_\alpha}(\xi) \leq C_1\times E(u)$ for some positive C_1. We can write $(-2,2)$ as a multiplication (from left to right) of the following transpositions

$$(-2,2) = (-2,-1)(-1,1)(1,2)(-1,1)(-1,-2)\,. \tag{33}$$

Setting $\xi_0 = \xi, \xi_1 = (\xi_0)_{-2,-1}, \xi_2 = (\xi_1)_{-1,1}, \xi_3 = (\xi_2)_{1,2}, \xi_4 = (\xi_3)_{-1,1}, \xi_5 = (\xi_4)_{-1,-2} = \xi_{-2,2}$ and applying the measure preserving transformations $\xi_i \to \xi$ yields:

$$
\begin{aligned}
\int \big(u(\xi_{-2,2}) - u(\xi)\big)^2 \, d\overline{\nu_\alpha}(\xi) &= \int \Big[\sum_{i=1}^{5} \big(u(\xi_i) - u(\xi_{i-1})\big)\Big]^2 \, d\overline{\nu_\alpha}(\xi) \\
&\le 5 \sum_{i=1}^{5} \int \big(u(\xi_i) - u(\xi_{i-1})\big)^2 \, d\overline{\nu_\alpha}(\xi) \\
&= 10 \int \big(u(\xi_{-2,-1}) - u(\xi)\big)^2 \, d\overline{\nu_\alpha}(\xi) + \\
&\quad 10 \int \big(u(\xi_{-1,1}) - u(\xi)\big)^2 \, d\overline{\nu_\alpha}(\xi) + \\
&\quad 5 \int \big(u(\xi_{1,2}) - u(\xi)\big)^2 \, d\overline{\nu_\alpha}(\xi) \\
&\le 40 D_{ex}(u) + C \times E(u) \le (C + 40) E(u) \qquad \square
\end{aligned}
$$

We conclude that

Lemma 24. $\left| \int \psi(\xi) u_\lambda(\xi) \, d\overline{\nu_\alpha}(\xi) \right| \le C_3(\alpha) \sqrt{E(u_\lambda)}$ holds for all $\lambda > 0$ with $C_3(\alpha) = C\sqrt{\alpha(1-\alpha)}$.

Proof. Fix $\lambda > 0$. We first note, that when viewing L and L_{ex} as operators on $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, clearly $\mathcal{D}(L) = \mathcal{D}(L_{ex})$ since $L - L_{ex} = L_{sh}$ which is a bounded operator. Since $u_\lambda \in C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$, by the core property of $\mathcal{C}$ relative to L_{ex} when viewed as an operator on $C\big(\{0,1\}^{\mathbb{Z}} \cap \{\xi : \xi(0) = 1\}\big)$ we can find a sequence $u_n \in \mathcal{C}$ such that $||u_n - u_\lambda||_\infty \to 0$ and $||L_{ex}u_n - L_{ex}u_\lambda||_\infty \to 0$ as $n \to \infty$. By Lemma 16,

$$
\begin{aligned}
D_{ex}(u_\lambda) &= \int \big(-L_{ex}u_\lambda\big)(\xi) u_\lambda(\xi) \, d\overline{\nu_\alpha}(\xi) \\
&= \lim_{n\to\infty} \int \big(-L_{ex}u_n\big)(\xi) u_n(\xi) \, d\overline{\nu_\alpha}(\xi) \\
&= \lim_{n\to\infty} D_{ex}(u_n)
\end{aligned}
$$

By bounded convergence,

$$
\begin{aligned}
&\frac{1}{8} \sum_{x\in\{\pm1,\pm2\}} \int (x + u_n(\tau_x\xi) - u_n(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi) \to \\
&\frac{1}{8} \sum_{x\in\{\pm1,\pm2\}} \int (x + u_\lambda(\tau_x\xi) - u_\lambda(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi)
\end{aligned}
$$

as $n \to \infty$. Thus,

$$
\begin{aligned}
E(u_\lambda) &= D_{ex}(u_\lambda) + \frac{1}{8} \sum_{x\in\{\pm1,\pm2\}} \int (x + u_\lambda(\tau_x\xi) - u_\lambda(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi) \\
&= \lim_{n\to\infty} D_{ex}(u_n) + \lim_{n\to\infty} \frac{1}{8} \sum_{x\in\{\pm1,\pm2\}} \int (x + u_n(\tau_x\xi) - u_n(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi) \\
&= \lim_{n\to\infty} \Big(D_{ex}(u_n) + \frac{1}{8} \sum_{x\in\{\pm1,\pm2\}} \int (x + u_n(\tau_x\xi) - u_n(\xi))^2 (1 - \xi(x)) \, d\overline{\nu_\alpha}(\xi) \Big) \\
&= \lim_{n\to\infty} E(u_n)
\end{aligned}
$$

Thus,

$$\begin{aligned}\left|\int \psi(\xi)u_\lambda(\xi)\,d\overline{\nu_\alpha}(\xi)\right| &= \lim_{n\to\infty}\left|\int \psi(\xi)u_n(\xi)\,d\overline{\nu_\alpha}(\xi)\right| \\ &\le C_3(\alpha)\lim_{n\to\infty}\sqrt{E(u_n)} \\ &= C_3(\alpha)\sqrt{E(u_\lambda)}\end{aligned}$$

□

In order to prove Theorem 1, we need the following results in section 4 in part III of Liggett ([21]). The results can be found in equations 4.26 and 4.49 and the paragraphs which precede and follow it, Proposition 4.1 and Theorems 4.39, 4.45 and 4.50. We already proved that the conditions of the theorem below holds by the symmetry of the drift and Lemmas 7, 8, 9, 20, 21 and 22.

Theorem 22. We assume that L and L_1 are Markov generators, that ξ_t is stationary and ergodic, that condition H_{-1} holds and that $\mathbb{E}\psi = 0$. Then the following holds:

(i) $X_t = \int_0^t \psi(\xi_s)\,ds + M_t$, where M_t is a square integrable martingale which satisfies that $\frac{M_t}{\sqrt{t}}$ converges in distribution to a Gaussian random variable with non-zero variance and zero mean.

(ii) $\int_0^t \psi(\xi_s)\,ds = N(t)+D(t)$, where $N(t)$ is a martingale which satisfies $\mathbb{E}|N(t)|^2 \le Ct$ and $\lim_{t\to\infty}\frac{\mathbb{E}|D(t)|^2}{t} = 0$.

(iii) $u_\lambda(\xi_t) - u_\lambda(\xi_0) = \int_0^t (Lu_\lambda)(\xi_s)\,ds + N_\lambda(t)$, where $N_\lambda(t)$ is a martingale which satisfies $\mathbb{E}N_\lambda^2(t) = 2tD(u_\lambda)$ and $N_\lambda(t)$ converges to $N(t)$ in $\mathcal{L}^2(\overline{\nu_\alpha})$ as $\lambda \downarrow 0$. In addition,

$$\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}N_\lambda^2(t) = \lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}\Big[u_\lambda(\xi_t) - u_\lambda(\xi_0)\Big]^2 = 2D(u_\lambda) \tag{34}$$

(iv) $\lim_{\lambda\downarrow 0}\lambda\int u_\lambda^2(\xi)\,d\overline{\nu_\alpha}(\xi) = 0$.

(v) $\frac{X_t}{\sqrt{t}}$ converges in distribution to a (possibly degenerate) Gaussian random variable with zero mean.

Before proving the main theorem we prove that $D(u_\lambda)$ satisfies the following property.

Lemma 25. $D(u_\lambda) \not\to 0$ as $\lambda \downarrow 0$.

Proof. Let $a(\xi) = \min\{i \ge 1 : \xi(i) = \xi(i+1) = 1\}$ so for all $x \in \{\pm 1, \pm 2\}$, $a(\tau_x\xi) - a(\xi) = -x$ holds for a.e. ξ with respect to $\overline{\nu_\alpha}$ and thus $-L_{sh}a = \psi$ a.s. holds. Since by Lemma 17, $D_{sh}(f) = \int f(-L_{sh}f)\,d\overline{\nu_\alpha}$ and by Lemma 12 $\int f(-L_{sh}g)\,d\overline{\nu_\alpha} = \int g(-L_{sh}f)\,d\overline{\nu_\alpha}$ we obtain

$$\begin{aligned}&\frac{1}{8}\sum_{x\in\{\pm1,\pm2\}}\int (x + u_\lambda(\tau_x\xi) - u_\lambda(\xi))^2(1-\xi(x))\,d\overline{\nu_\alpha}(\xi) \\ &= \frac{1}{8}\sum_{x\in\{\pm1,\pm2\}}\int (u_\lambda(\tau_x\xi) - a(\tau_x\xi) - u_\lambda(\xi) + a(\xi))^2(1-\xi(x))\,d\overline{\nu_\alpha}(\xi) = D_{sh}(u_\lambda - a)\end{aligned}$$

and thus

$$\begin{aligned}E(u_\lambda) &= D_{ex}(u_\lambda) + D_{sh}(u_\lambda - a) \\ &= D_{ex}(u_\lambda) + \int (u_\lambda - a)(-L_{sh})(u_\lambda - a)\,d\overline{\nu_\alpha} \\ &= D_{ex}(u_\lambda) + D_{sh}(u_\lambda) + D_{sh}(a) - \int u_\lambda(-L_{sh}a)\,d\overline{\nu_\alpha} - \int a(-L_{sh}u_\lambda)\,d\overline{\nu_\alpha} \\ &= D(u_\lambda) + D_{sh}(a) - 2\int \psi u_\lambda\,d\overline{\nu_\alpha}\end{aligned}$$

Plugging in $\psi = \lambda u_\lambda - Lu_\lambda$ yields

$$\begin{aligned} E(u_\lambda) &= D(u_\lambda) + D_{sh}(a) - 2\int (\lambda u_\lambda - Lu_\lambda)u_\lambda \, d\overline{\nu_\alpha} \\ &= D(u_\lambda) + D_{sh}(a) - 2\lambda \int u_\lambda^2 \, d\overline{\nu_\alpha} - 2D(u_\lambda) \\ &= D_{sh}(a) - D(u_\lambda) - 2\lambda \int u_\lambda^2 \, d\overline{\nu_\alpha} \end{aligned} \tag{35}$$

We estimate $D_{sh}(a)$. We plug in $\psi = \lambda u_\lambda - Lu_\lambda$

$$\begin{aligned} D_{sh}(a) &= \int a(-L_{sh}a) \, d\overline{\nu_\alpha} \\ &= \int \psi a \, d\overline{\nu_\alpha} \\ &= \int (\psi - \lambda u_\lambda) a \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \\ &= \int a(-Lu_\lambda) \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \end{aligned}$$

Writing $L = L_{ex} + L_{sh}$ and applying once again $\int f(-L_{sh}g) \, d\overline{\nu_\alpha} = \int g(-L_{sh}f) \, d\overline{\nu_\alpha}$ yields:

$$\begin{aligned} D_{sh}(a) &= \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \int a(-L_{sh}u_\lambda) \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \\ &= \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \int u_\lambda(-L_{sh}a) \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \\ &= \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \int \psi u_\lambda \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \end{aligned}$$

Plugging in $\psi = \lambda u_\lambda - Lu_\lambda$ yields

$$\begin{aligned} D_{sh}(a) &= \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \int (\lambda u_\lambda - Lu_\lambda)u_\lambda \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} \\ &= D(u_\lambda) + \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} + \lambda \int u_\lambda^2 \, d\overline{\nu_\alpha} \end{aligned}$$

Thus, plugging this into equation (35) yields

$$E(u_\lambda) = \int a(-L_{ex}u_\lambda) \, d\overline{\nu_\alpha} + \lambda \int u_\lambda a \, d\overline{\nu_\alpha} - \lambda \int u_\lambda^2 \, d\overline{\nu_\alpha}$$

By Lemma 11, for all $f, g \in \mathcal{D}(L_{ex})$ we have $\int \big(L_{ex}f\big) g \, d\overline{\nu_\alpha} = \int f \big(L_{ex}g\big) \, d\overline{\nu_\alpha}$ and by Lemma 16, for all $a \in \mathcal{D}(L_{ex})$ we have $0 \leq D_{ex}(a) = \int \big(-L_{ex}a\big)(\xi)a(\xi) \, d\overline{\nu_\alpha}(\xi)$, so we can define a semi inner product on $\mathcal{D}(L_{ex})$ via $\langle a, b\rangle_{ex} = \int a(-L_{ex}b) \, d\overline{\nu_\alpha}$. By applying Cauchy-Schwartz and applying part (iv) in Theorem 22, we conclude that

$$E(u_\lambda) \leq \sqrt{D_{ex}(a)}\sqrt{D_{ex}(u_\lambda)} + o(1) \tag{36}$$

as $\lambda \downarrow 0$. Thus, if we assume that $D(u_\lambda) \to 0$ as $\lambda \downarrow 0$ then by equation (36)

$$E(u_\lambda) \to 0$$

as $\lambda \downarrow 0$ and by equation (35), part (iv) in Theorem 22 and by the definition of $D_{sh}(a)$ after Lemma 12, together with the above fact that $a(\eta_x \xi) - a(\xi) = -x$ for $x \in \{\pm 1, \pm 2\}$

$$E(u_\lambda) \to D_{sh}(a) \neq 0$$

as $\lambda \downarrow 0$ so thus the assumption cannot hold completing the proof. $\square$

Theorem 1. The position of the tagged particle, X_t, satisfies $X_t/\sqrt{t}$ converges in distribution to a normal random variable with non-zero variance and zero mean.

Proof. The proof is similar to the proof of Theorem 4.55 in section 4 in part III of Liggett ([21]). By parts (i) and (ii) in Theorem 22, we can write $X_t = N(t) + M_t + D(t)$. By part (v) of Theorem 22 $X_t/\sqrt{t}$ converges in distribution to a mean zero normal random variable. Thus, it suffices to show that $\frac{1}{t}\mathbb{E}(X_t)^2 = \frac{1}{t}\mathbb{E}(N(t) + M_t + D(t))^2 \geq C$ for some positive constant C for t sufficiently large which will follow from proving $\frac{\mathbb{E}(N(t)+M_t)^2}{t} \geq C$ since $\lim_{t\to\infty} \frac{\mathbb{E}|D(t)|^2}{t} = 0$. Since $N(t) + M_t$ is a martingale, it has orthogonal increments and thus, using also that η_t is stationary, we obtain that $\mathbb{E}(N(t)+M_t)^2 = t\mathbb{E}(N(1)+M_1)^2$. Thus, it suffices to prove that $\mathbb{E}(N(1)+M_1)^2 > 0$. For all $\lambda > 0$ by equation (31), $D(u_\lambda) \leq \int \psi u_\lambda \, d\overline{\nu_\alpha}$ and by Lemma 24, $\int \psi u_\lambda \, d\overline{\nu_\alpha} \leq C\sqrt{E(u_\lambda)}$ so

$$D(u_\lambda) \leq C\sqrt{E(u_\lambda)} \tag{37}$$

Lemma 25 tells us that $D(u_\lambda) \not\to 0$ as $\lambda \downarrow 0$. By equation (37), $E(u_\lambda) \not\to 0$ as $\lambda \downarrow 0$. For $g_\lambda(x, \xi) = x + u_\lambda(\xi)$ the following holds by adding the two equations in parts (i) and (iii) in Theorem 22 and recalling that $\psi + Lu_\lambda = \lambda u_\lambda$ and noting that $X_0 = 0$:

$$g_\lambda(X_t, \xi_t) - g_\lambda(X_0, \xi_0) = \int_0^t \lambda u_\lambda(\xi_s) \, ds + N_\lambda(t) + M_t$$

Since by part (iv) in Theorem 22, $\lambda \int u_\lambda^2 \, d\overline{\nu_\alpha}$ is bounded for $\lambda > 0$ bounded from ∞ we obtain by Cauchy-Schwartz, noting that by stationarity

$$\mathbb{E}\left[u_\lambda^2(\xi_s)\right] = \int \mathbb{E}^\xi(u_\lambda^2)(\xi_s) \, d\overline{\nu_\alpha}(\xi) = \int (S(s)u_\lambda^2)(\xi) \, d\overline{\nu_\alpha}(\xi) = \int u_\lambda^2(\xi) \, d\left[\overline{\nu_\alpha} S(s)\right](\xi) = \int u_\lambda^2(\xi) \, d\overline{\nu_\alpha}(\xi)$$

(here $S(s)$ is the semigroup generated by L):

$$\mathbb{E}\left[\int_0^t \lambda u_\lambda(\xi_s) \, ds\right]^2 \leq \lambda^2 t \mathbb{E}\left[\int_0^t u_\lambda^2(\xi_s) \, ds\right] = \lambda^2 t^2 \int u_\lambda^2 \, d\overline{\nu_\alpha} \leq C\lambda t^2 \tag{38}$$

Thus:

$$\begin{aligned} \mathbb{E}\left[N_\lambda(1) + M_1\right]^2 &= \frac{\mathbb{E}\left[N_\lambda(t) + M_t\right]^2}{t} \\ &= \frac{\mathbb{E}\left[g_\lambda(X_t, \xi_t) - g_\lambda(X_0, \xi_0)\right]^2}{t} + \frac{\mathbb{E}\left[\int_0^t \lambda u_\lambda(\xi_s) \, ds\right]^2}{t} \\ &\quad - \frac{2\mathbb{E}\left[\left(g_\lambda(X_t, \xi_t) - g_\lambda(X_0, \xi_0)\right)\int_0^t \lambda u_\lambda(\xi_s) \, ds\right]}{t} \end{aligned} \tag{39}$$

So by equation (38),

$$\lim_{t\downarrow 0} \frac{\mathbb{E}\Big[\int_0^t \lambda u_\lambda(\xi_s)\,ds\Big]^2}{t} = 0 \tag{40}$$

and similarly,

$$\begin{aligned}
\frac{2\Big|\mathbb{E}\Big[\Big(g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big)\int_0^t \lambda u_\lambda(\xi_s)\,ds\Big]\Big|}{t} &\leq \frac{2\Big(\mathbb{E}\Big[g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big]^2\Big)^{1/2}}{t}\times \\
&\Big(\mathbb{E}\Big[\int_0^t \lambda u_\lambda(\xi_s)\,ds\Big]^2\Big)^{1/2} \\
&\leq 2\sqrt{C}\Big(\mathbb{E}\Big[g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big]^2\Big)^{1/2} \\
&\leq 2\sqrt{C}\Big(2\mathbb{E}\Big[\int_0^t \lambda u_\lambda(\xi_s)\,ds\Big]^2 + 2\mathbb{E}\Big[N_\lambda(t)+M_t\Big]^2\Big)^{1/2} \\
&\leq 2\sqrt{C}\Big(2Ct^2 + 2t\mathbb{E}\Big[N_\lambda(1)+M_1\Big]^2\Big)^{1/2} \to 0
\end{aligned}$$

as $t \downarrow 0$ so

$$\lim_{t\downarrow 0} -\frac{2\mathbb{E}\Big[\Big(g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big)\int_0^t \lambda u_\lambda(\xi_s)\,ds\Big]}{t} = 0. \tag{41}$$

We still need to deal with the term $\frac{1}{t}\mathbb{E}\Big[g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big]^2$. We recall that $X_0 = 0$.

$$\begin{aligned}
\frac{1}{t}\mathbb{E}\Big[g_\lambda(X_t,\xi_t)-g_\lambda(X_0,\xi_0)\Big]^2 &= \frac{1}{t}\mathbb{E}\Big[X_t + u_\lambda(\xi_t) - u_\lambda(\xi_0)\Big]^2 \\
&= \mathbb{E}\Big[\frac{X_t^2}{t} + 2\frac{X_t u_\lambda(\xi_t)}{t} - 2u_\lambda(\xi_0)\frac{X_t}{t}\Big] + \frac{1}{t}\mathbb{E}\Big[u_\lambda(\xi_t)-u_\lambda(\xi_0)\Big]^2 \\
&= \int \mathbb{E}^{(X_0,\xi_0)}\Big[\frac{X_t^2}{t} + 2\frac{X_t u_\lambda(\xi_t)}{t} - 2u_\lambda(\xi_0)\frac{X_t}{t}\Big]\,d\overline{\nu_\alpha}(\xi_0) + \frac{1}{t}\mathbb{E}\Big[u_\lambda(\xi_t)-u_\lambda(\xi_0)\Big]^2 \\
&= \int \Big[\frac{1}{t}\mathbb{E}^{(X_0,\xi_0)}X_t^2 + 2\times\frac{1}{t}\mathbb{E}^{(X_0,\xi_0)}(X_t u_\lambda(\xi_t)) - 2u_\lambda(\xi_0)\times\frac{1}{t}\mathbb{E}^{(X_0,\xi_0)}X_t\Big]\,d\overline{\nu_\alpha}(\xi_0) \\
&+ \frac{1}{t}\mathbb{E}\Big[u_\lambda(\xi_t)-u_\lambda(\xi_0)\Big]^2 \\
&\to \int \Big[(L_1X^2)(0,\xi) + 2(L_1Xu_\lambda)(0,\xi) - 2u_\lambda(\xi)(L_1X)(0,\xi)\Big]\,d\overline{\nu_\alpha}(\xi) + 2D(u_\lambda)
\end{aligned}$$

as $t \downarrow 0$ by equation (34). We note that the functions $f_1(x,\xi) = x^2$, $f_2(x,\xi) = xu_\lambda(\xi)$ and $f_3(x,\xi) = x$ do not belong to the domain of L_1 as they are unbounded functions, so an approximation argument is required in order to justify the last step. When we applied L_1 to these functions we meant plugging in the functions into the formula of L_1 noting that since the functions f_1 and f_3 do not depend on the movement of the untagged particles clearly $L_{ex}f_1 = L_{ex}f_3 = 0$ and similarly $(L_{ex}f_2)(x,\xi) = xL_{ex}(u_\lambda)$ so all the infinite sums converge and thus the integrals can be evaluated. We provide the full approximation argument for f_1 (the other functions are dealt with in a similar way). Let f_1^n be a non-negative sequence of functions belonging to the domain of L_1 which depend only on x (i.e. for all x and η and all n, $f_1^n(x,\eta) = f_1^n(x)$) which satisfy for each n $f_1^n(x,\eta) = x^2$ whenever $|x| \leq n$ and also $f_1^n \leq f_1^{n+1}$ (i.e. the sequence is an increasing sequence) and $f_1^n(x,\eta) \leq x^2$. Let $S(t)$ be the semigroup generated by L_1, i.e. $(S(t)f)(X_0,\xi_0) = \mathbb{E}^{(X_0,\xi_0)} f(X_t,\xi_t)$. By monotone convergence, for each ξ_0, $(S(t)f_1^n)(0,\xi_0) \nearrow \mathbb{E}^{(0,\xi_0)} X_t^2$ as $n \to \infty$. Thus, by Fatou's lemma

$$\begin{aligned}\int \left| \frac{1}{t}\mathbb{E}^{(0,\xi)} X_t^2 - (L_1 X^2)(0,\xi) \right| d\overline{\nu_\alpha}(\xi) &= \int \lim_{n\to\infty} \left| \frac{1}{t}(S(t)f_1^n)(0,\xi) - (L_1X^2)(0,\xi) \right| d\overline{\nu_\alpha}(\xi) \\ &\leq \liminf_{n\to\infty} \int \left| \frac{1}{t}(S(t)f_1^n)(0,\xi) - (L_1X^2)(0,\xi) \right| d\overline{\nu_\alpha}(\xi) \\ &\leq \liminf_{n\to\infty} \sup_\xi \left| \frac{1}{t}(S(t)f_1^n)(0,\xi) - (L_1X^2)(0,\xi) \right|\end{aligned}$$

The exclusion process for our model starting from (x,η) can be obtained by placing independent Poisson clocks with rate 1 on $\mathbb{Z}$. When the clock at the origin rings (x,η) moves with probability $1/4$ to $(x+i, \tau_i\eta)$ if $\eta(i) = 0$ for $i \in \{\pm 1, \pm 2\}$ and when the clock at position $j \neq 0$ rings then (x,η) moves to $(x, \eta_{j,j+i})$ with probability $1/2$ for $i \in \{\pm 1\}$ if $j+i \neq 0$ and $\eta(j+1) = 0$. Let A_t denote the number of times the Poisson clock at the origin rang until time t and let A_t^i denote the numbr of times the Poisson clock at i rang until time t. Thus,

$$\begin{aligned}(S(t)f_1^n)(0,\xi) = &\sum_{k\geq 0, k\neq 1} \mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = k \right] \mathbb{P}[A_t = k] \\ &+ \mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i = 0 \right] \mathbb{P}\left[A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i = 0 \right] \\ &+ \mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i > 0 \right] \mathbb{P}\left[A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i > 0 \right].\end{aligned}$$

Since for each $n \geq 2$ and $k \geq 2$ the following inequalities hold

$$\begin{aligned}&\mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = 0 \right] = 0 \\ &\mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i = 0 \right] = (L_1X^2)(0,\xi) \\ &\left| \mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = k \right] \right| \leq 4k^2 \\ &\mathbb{P}[A_t = k] = e^{-t}\frac{t^k}{k!} \\ &\mathbb{P}\left[A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i = 0 \right] = e^{-6t}e^{-t}t \\ &\mathbb{P}\left[A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i > 0 \right] = (1-e^{-6t})e^{-t}t \\ &\left| \mathbb{E}^{(0,\xi)}\left[f_1^n(X_t,\xi_t) \middle| A_t = 1, \sum_{\{i\in \pm 1,\pm 2,\pm 3\}} A_t^i > 0 \right] \right| \leq 4\end{aligned}$$

we conclude that for all ξ and all $n \geq 2$ and all $t > 0$, $\left| \frac{1}{t}(S(t)f_1^n)(0,\xi) - (L_1X^2)(0,\xi) \right| \leq Ct$ where C is independent of n and thus $\lim_{t\downarrow 0} \int \frac{1}{t}\mathbb{E}^{(0,\xi)} X_t^2 \, d\overline{\nu_\alpha}(\xi) = \int (L_1X^2)(0,\xi)\, d\overline{\nu_\alpha}(\xi)$ which completes the proof for f_1. We now return to the expression we obtained for $\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}\Big[g_\lambda(X_t,\xi_t) - g_\lambda(X_0,\xi_0) \Big]^2$ and we first deal with its last two terms. Since $(L_1X)(0,\xi) = \psi(\xi)$ we obtain

$$\int \left[-2u_\lambda(\xi)(L_1X)(0,\xi) \right] d\overline{\nu_\alpha}(\xi) + 2D(u_\lambda) = -2\int u_\lambda(\xi)\psi(\xi)\, d\overline{\nu_\alpha}(\xi) + 2D(u_\lambda) \tag{42}$$

We now deal with the first two terms. First we note that, by applying Lemma 12 to the functions $f = u_\lambda^2$ and $g = 1$, we obtain that $\int L_{sh} u_\lambda^2 \, d\overline{\nu} = 0$. Plugging in the formulas for L_1X^2 and L_1Xu_λ yields

$$\begin{aligned}
&\int \left[(L_1X^2)(0,\xi) + 2(L_1Xu_\lambda)(0,\xi) \right] d\overline{\nu_\alpha}(\xi) \\
=&\int \left[\frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} i^2(1-\xi(i)) + \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} 2iu_\lambda(\tau_i\xi)(1-\xi(i)) \right] d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\tau_i\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\xi) + u_\lambda^2(\xi) - u_\lambda^2(\tau_i\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) - \int (L_{sh}u_\lambda^2)(\xi)\, d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) .
\end{aligned}$$

If we write

$$i + u_\lambda(\tau_i\xi) = i + u_\lambda(\tau_i\xi) - u_\lambda(\xi) + u_\lambda(\xi)$$

then

$$\begin{aligned}
(i + u_\lambda(\tau_i\xi))^2 = &(i + u_\lambda(\tau_i\xi) - u_\lambda(\xi))^2 + u_\lambda^2(\xi) \\
&+ 2u_\lambda(\xi)(i + u_\lambda(\tau_i\xi) - u_\lambda(\xi))
\end{aligned}$$

and thus, recalling that $\psi(\xi) = \frac{1}{4}\sum_{i\in\{\pm1,\pm2\}} i(1-\xi(i))$ and by the definition of $E(a)$ which appears after Lemma 12 and by Lemma 17 we obtain:

$$\begin{aligned}
&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi) - u_\lambda(\xi))^2 + 2u_\lambda(\xi)(i + u_\lambda(\tau_i\xi) - u_\lambda(\xi)) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) \\
=&\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} (i + u_\lambda(\tau_i\xi) - u_\lambda(\xi))^2 (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) + 2\int u_\lambda(\xi)\psi(\xi)\, d\overline{\nu_\alpha} + 2\int u_\lambda(\xi)L_{sh}(\xi)\, d\overline{\nu_\alpha} \\
=&2E(u_\lambda) - 2D_{ex}(u_\lambda) + 2\int u_\lambda(\xi)\psi(\xi)\, d\overline{\nu_\alpha} - 2D_{sh}(u_\lambda)
\end{aligned}$$

so

$$\int \frac{1}{4} \sum_{i\in\{\pm1,\pm2\}} \left[(i + u_\lambda(\tau_i\xi))^2 - u_\lambda^2(\xi) \right] (1-\xi(i))\, d\overline{\nu_\alpha}(\xi) = 2E(u_\lambda) - 2D(u_\lambda) + 2\int u_\lambda(\xi)\psi(\xi)\, d\overline{\nu_\alpha} \tag{43}$$

By adding equations (42) and (43) we conclude that

$$\int \Big[(L_1 X^2)(0,\xi) + 2(L_1 X u_\lambda)(0,\xi) - 2u_\lambda(\xi)(L_1 X)(0,\xi) \Big] d\overline{\nu_\alpha}(\xi) + 2D(u_\lambda) = 2E(u_\lambda)$$

and thus

$$\lim_{t \downarrow 0} \frac{\mathbb{E}\Big[g_\lambda(X_t, \xi_t) - g_\lambda(X_0, \xi_0) \Big]^2}{t} = 2E(u_\lambda) \tag{44}$$

Thus, by plugging in equations (40), (41) and (44) into equation (39)

$$\mathbb{E}\Big[N_\lambda(1) + M_1 \Big]^2 = 2E(u_\lambda)$$

Since $N_\lambda(1) \to N(1)$ in $\mathcal{L}^2(\overline{\nu_\alpha})$ as $\lambda \downarrow 0$ by part (iii) in Theorem 22 we obtain:

$$\mathbb{E}\Big[N(1) + M_1 \Big]^2 = 2 \lim_{\lambda \downarrow 0} E(u_\lambda) > 0$$

as observed after equation (37) completing the proof. □

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